

The Excel FFT Function v1.1
P. T. Debevec
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The discrete Fourier transform may be used to identify periodic structures in time series data. Suppose that a physical process is represented by the function of time, $h(t)$. The function is sampled at N times, $t_k = k\Delta t$ where $k = 0, 1, 2, \dots, N-1$. From these N measurements, h_k , N complex amplitudes, H_n , are determined which satisfy the N equations

$$H_n = \sum_{k=0}^{N-1} h_k e^{ik \frac{2\pi n}{N}}.$$

The sampled function then has the discrete Fourier expansion

$$h_k = \frac{1}{N} \sum_{n=0}^{N-1} H_n e^{-in \frac{2\pi k}{N}}.$$

This equation can be cast in familiar form with $2\pi k/N = (2\pi/T_o)k(T_o/N) = \omega_o k \Delta t = \omega_o t_k$

$$h_k = \frac{1}{N} \sum_{n=0}^{N-1} H_n e^{-in\omega_o t_k}.$$

The right-hand side is the discrete analogue to the complex form of the Fourier expansion

$$h(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega_o t}$$

where the complex coefficients, c_n , are given by

$$c_n = \frac{1}{T_o} \int_0^{T_o} h(t) e^{-in\omega_o t} dt.$$

The Excel data analysis package has a Fourier analysis routine which calculates the complex coefficients, H_n , from the time series data, h_k . The routine requires that the number of samples in the time series data be a power of 2, i.e. $N = 2^m$. The example in this note uses $N = 2048$. The Excel function is not well documented, but it is straightforward to use. This note describes the Excel worksheet, `Fourier_example.xls`, which is in the Physics 401 web site under Tutorials and Lectures, Experiment 10.

The time series data in this example are obtained from sampling a function describing the free decay of a torsion oscillator for time $t > t_o$,

$$\theta(t) = Ae^{-a(t-t_o)} \sin(2\pi f(t-t_o)).$$

The function is calculated in time steps of 0.020 s, which corresponds to sampling rate of 50.0 Hz. The time series data are shown in the Fig. 1 below.

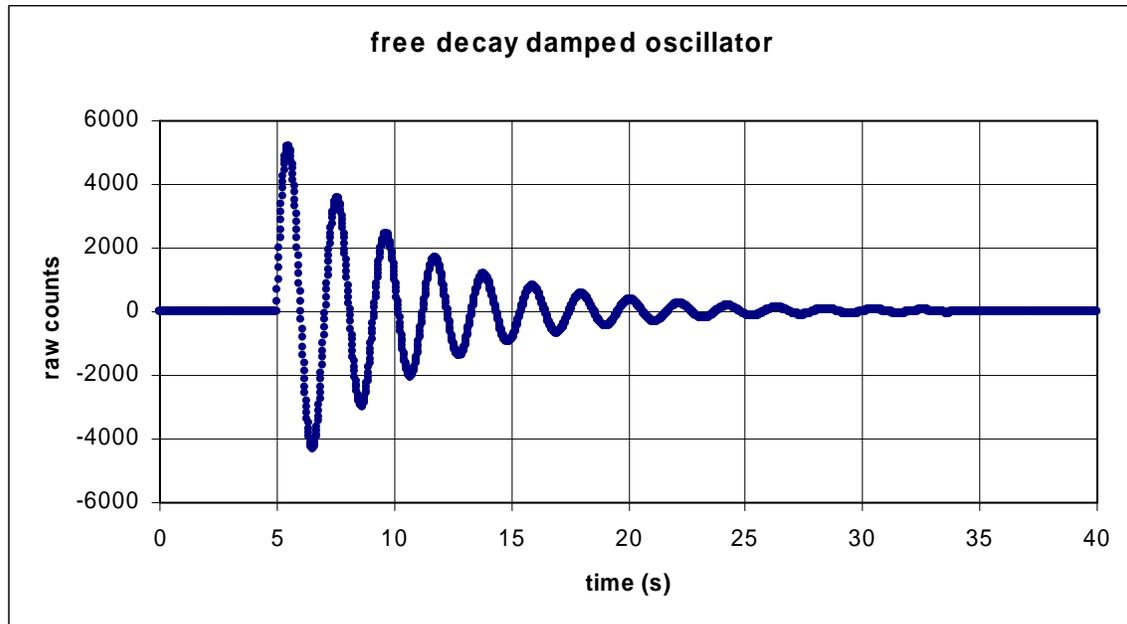


Fig. 1 Plot of time series data – decay of torsion oscillator

The data occupy cells B3 to B2050 in the data worksheet of the workbook. Click on Tools in the Excel menu bar, and select Data Analysis. In Data Analysis select Fourier Analysis, and a simple dialog box appears. The dialog box is shown below in Fig. 2.

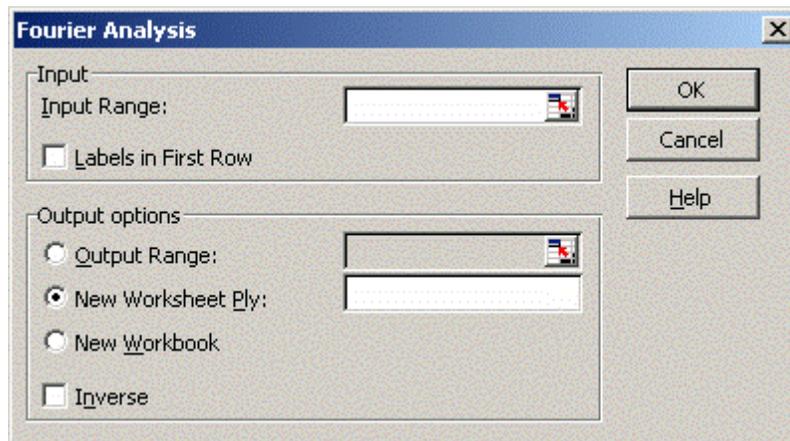


Fig. 2 Fourier Analysis dialog box

For Input Range enter \$B\$3:\$B\$2050, the location of the time series data, and for output range enter a convenient place on the worksheet, for example, \$J\$3:\$J\$2050. After selecting the OK button, Excel returns in column J the complex coefficients, 2048 of them. A portion of column J is shown in Fig. 3 below.

	I	J	K	L	M
1					
2	frequency	Excel FFT output			
3	0.0000	94461.266925616			
4	0.0244	67786.9828448472-66143.8573465778i			
5	0.0488	2343.00833422099-95427.9827107803i			
6	0.0732	-65824.8267423328-70853.106743236i			
7	0.0977	-98386.8029821029-4795.30767174509i			

Fig. 3 Portion of Excel worksheet showing FFT output

Excel insists in displaying 15 digits for both the real and imaginary part of the coefficient, and thus the output occupies consider space. Examination of the column of complex numbers shows that the first entry (in cell J3) and the 1025th entry (in cell J1027) are real. These cells represent the coefficients for the zero frequency and the folding frequency, f_c . Since the data were sampled at 50.0 Hz, the folding frequency is $\frac{1}{2} \times 50 \text{ Hz} = 25 \text{ Hz}$. Cells J3 through J1027 represent the complex coefficients for $N/2 + 1$ frequencies from 0 Hz to 25 Hz. The frequency increment is then $\Delta f = f_c / N$. It is convenient to make a column of frequencies next to the column of coefficients. This column includes cells I3 to I1027. The Fourier Analysis routine operates only on the time series data. The correspondence between frequency and complex coefficient must be calculated independently. The 1023 entries from cell J1028 to cell J2050 are identical to the 1023 entries from cell J1026 to J4. These entries contain no additional information, and they could have been omitted from the display. The Fourier Analysis of N time series data yields the complex coefficients for only $N/2 + 1$ frequencies.

The power in each frequency bin is proportional to the square of the modulus of the complex coefficient. The constant of proportionality adopted in this note has the property that the mean squared amplitude of the time series data is equal to the total power in the frequency domain as shown below. Excel has a number of functions for complex arguments. Multiplication of two complex numbers is accomplished with the function IMPRODUCT(z_1, z_2). Multiplication of the coefficients in column J by $1/N$ produces the normalized coefficients in column N. The absolute value of a complex number is accomplished with the function IMABS(z). Applying the IMABS function to the normalized coefficients in column N and multiplying by a factor of $\sqrt{2}$ produces the magnitude of the Fourier coefficients in column P. The magnitude of the Fourier coefficients as a function of frequency is shown in Fig. 4 below. The maximum value occurs in the frequency bin at 0.4883 Hz.

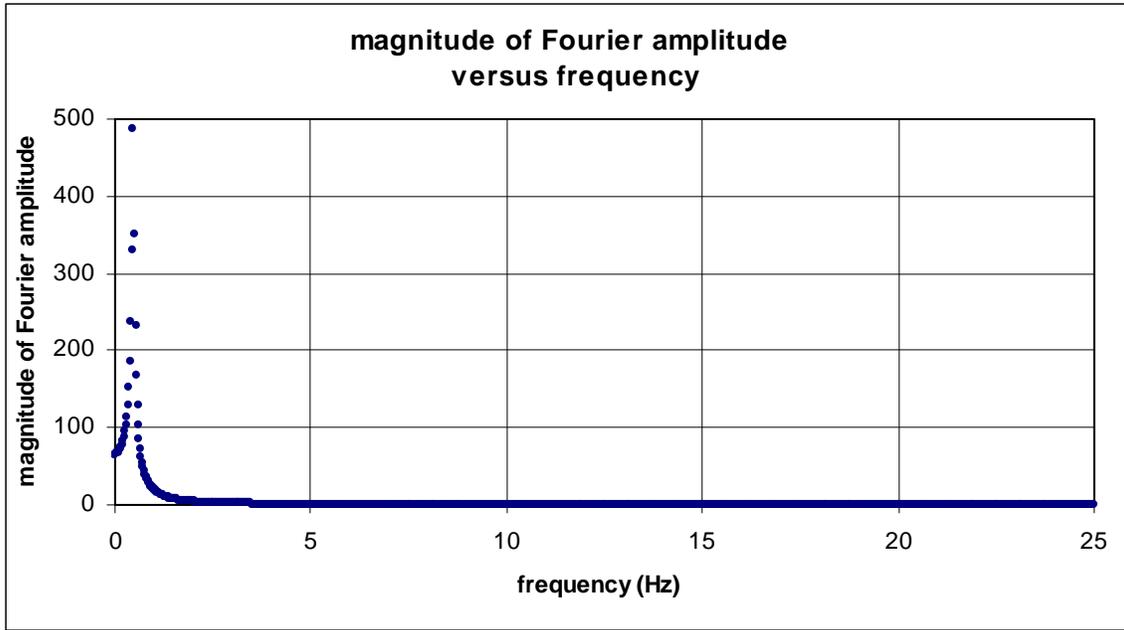


Fig. 4 Magnitude of normalized Fourier coefficient versus frequency

Finally, the power at each frequency (except zero frequency) is the square of the magnitude. The power is calculated in column Q and shown in Fig. 5 below. The power at zero frequency is the square of the magnitude divided by 2. This factor of 2 also appears in non-discrete Fourier analysis. This power distribution is characteristic of exponential decay.

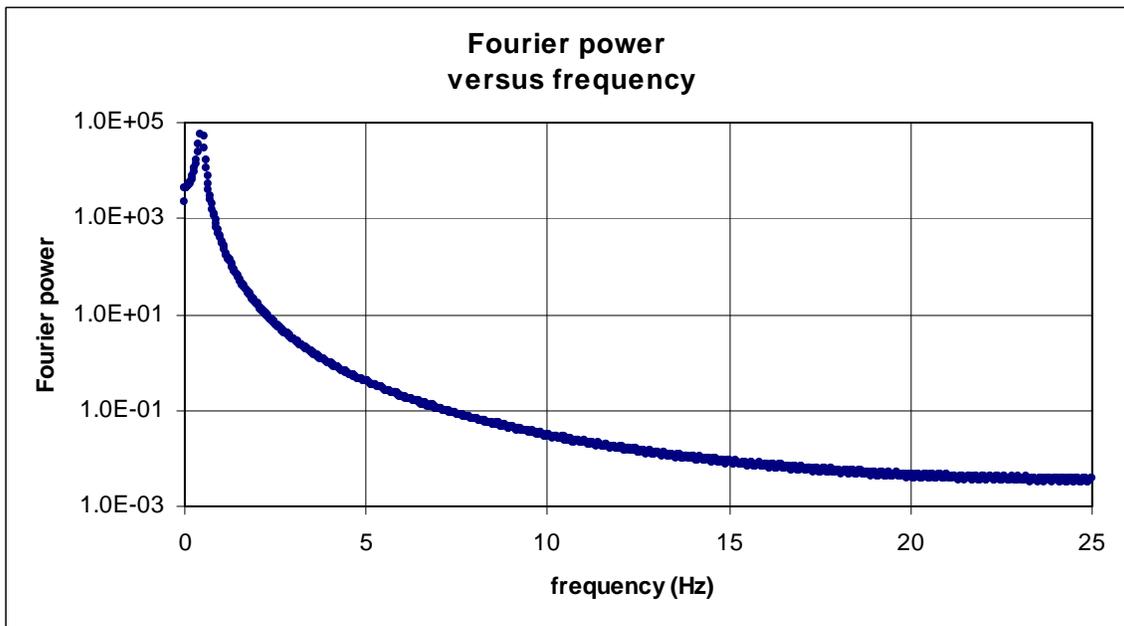


Fig. 5 Fourier power versus frequency

The normalization adopted in this note has a convenient property. Define the mean squared amplitude of the time series data as

$$\text{mean squared amplitude} = \frac{1}{N} \sum_{k=0}^{N-1} [\theta(t_k)]^2.$$

The square of the amplitude is calculated in column C. The mean squared amplitude is calculated in cell F19. For these data it has the value of 1105488.754. A pure sine wave with an amplitude of $\sqrt{2}$, for which an integral number of periods occur in the total sampled time, has a mean squared amplitude of unity. Thus the mean squared amplitude has the same role as the rms value of a periodic function. The total Fourier power, the sum of the entries in column Q, is calculated in cell G19. For these data it also has the value of 1105488.754, because of the normalization. If only the relative power in different frequency bins is needed, then the normalization is not required. The relative power is given by the square of the modulus of the Excel output directly.