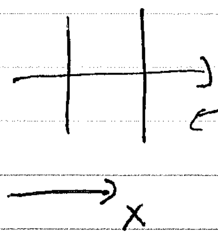
Van de Hulst's Anomalous Diffraction Theory ADT

Aside $\quad k_r = \frac{2\pi}{\lambda_0} n_r \quad k_i = \frac{2\pi}{\lambda_0} n_i \quad , \lambda_0 = c/f$



← Plane wave in media $n_r + i n_i$ is $\Psi = \tilde{A}\, e^{(i k_r x - k_i x)}$

Notation: I = Irradiance

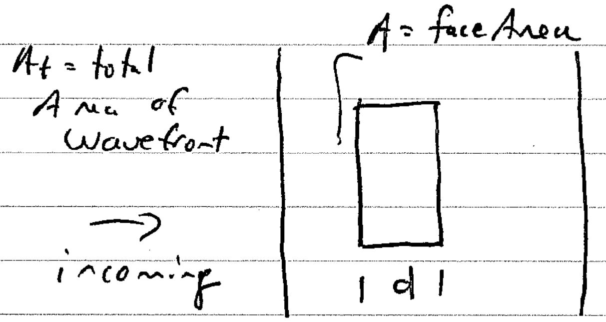
n_r = real part of refractive index

n_i = imag " " " $\qquad n = n_r + i n_i$

E = Electric field $\quad , \quad I \propto E E^*$

k = Propagation "Constant" $= \frac{n\omega}{c} = \frac{n_r \omega}{c} + i \frac{n_i \omega}{c}$

$\omega = 2\pi f$



ADT Theory $\quad n_r \approx 1$, $n_i \ll 1$

Light that goes through Particle has a phase shift relative to light that doesn't.

Total transmitted E field:

$$E_t = E_0 e^{ikd} \frac{A}{A_t} + E_0 e^{ik_0 d} \frac{(A_t - A)}{A_t}$$

Through Particle $\qquad$ Around Particle

Total transmitted Irradiance: $I_t = E_t E_t^*$

factor a bit

$$I_t = \left| \frac{E_0 e^{ik_0 d} A}{A_t} \left[1 + \frac{(A_t - A)}{A} e^{i(k-k_0)d} \right] \right|^2$$

$$= \frac{E_0^2 A^2}{A_t^2} \left(1 + \frac{A_t - A}{A} e^{i(k-k_0)d} \right) \left(1 + \frac{A_t - A}{A} e^{-i(k-k_0)d} \right)$$

$\beta = \frac{2\pi n_i}{\lambda_0}$

$$= \frac{E_0^2 A^2}{A_t^2} \left(1 + 2 \frac{(A_t - A)}{A} \cos[(k-k_0)d]\, e^{-\beta d} + \left(\frac{A_t - A}{A} \right)^2 \right)$$

$$2\cos[(k-k_0)d] = e^{i(k-k_0)d} + e^{-i(k-k_0)d}$$

ADT theory Continued

or

$$I_t = E_0^2\left[\frac{A^2}{A_t^2} + \frac{2(A_t-A)A}{A_t^2}\cos[(k-k_0)d]\,e^{-\beta d} + 1 - \frac{2A}{A_t} + \frac{A^2}{A_t^2}\right]$$

Since $A_t \gg A$, keep only terms like A/A_t

Find that

$$I_t = I_0 - \frac{I_0\,\sigma_{ext}}{A_t}$$

So the power in the beam is

$$P_t = I_t A_t = A_t I_0 - I_0\,\sigma_{ext}$$

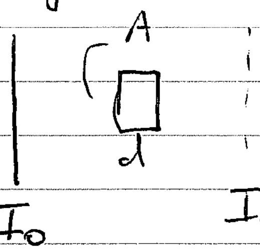
Extinction

$$\boxed{\sigma_{ext} = 2A\left[1 - e^{-\frac{2\pi\Lambda_i d}{\lambda}}\cos\left((\Lambda_r-1)\frac{2\pi d}{\lambda}\right)\right]}$$

Embodies interference of the forward going beam as the cause for extinction when $\Lambda_r \gtrsim 1$. Destructive interference in the forward direction means energy is going somewhere else.

Theory not so good for $|\Lambda_r - 1| \gtrsim 0.2$

Absorption



$$I_t = \left(\frac{A_t - A}{A_t}\right)I_0 + I_0\frac{A}{A_t}e^{-\frac{4\pi\Lambda_i d}{\lambda}}$$

In the particle $I_t = E E^* = |e^{-\beta d} e^{ikd}|$

$= e^{-2\beta d}$ $\quad \beta = 2\pi\Lambda_i/\lambda$

Similar Argument gives

$$\boxed{\sigma_{abs} = A\left(1 - e^{-\frac{4\pi\Lambda_i d}{\lambda}}\right)}$$

good for $|\Lambda_r - 1| \gtrsim 0$

Limits $\frac{4\pi\Lambda_i d}{\lambda} \gg 1$

$\sigma_{abs} \doteq A$

$\ll 1$

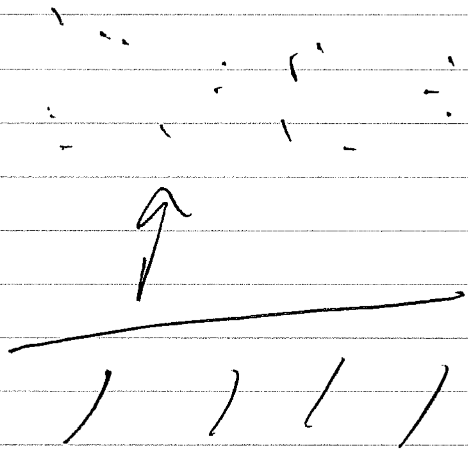
Analytic

$\sigma_{abs} \doteq \frac{4\pi D^3}{\lambda}\Lambda_i$

Volume limit

Application

Cloud Emissivity : (Cirrus, for $|n_r - 1| \simeq 0$



$$t = e^{-\sigma_{abs} N L}$$

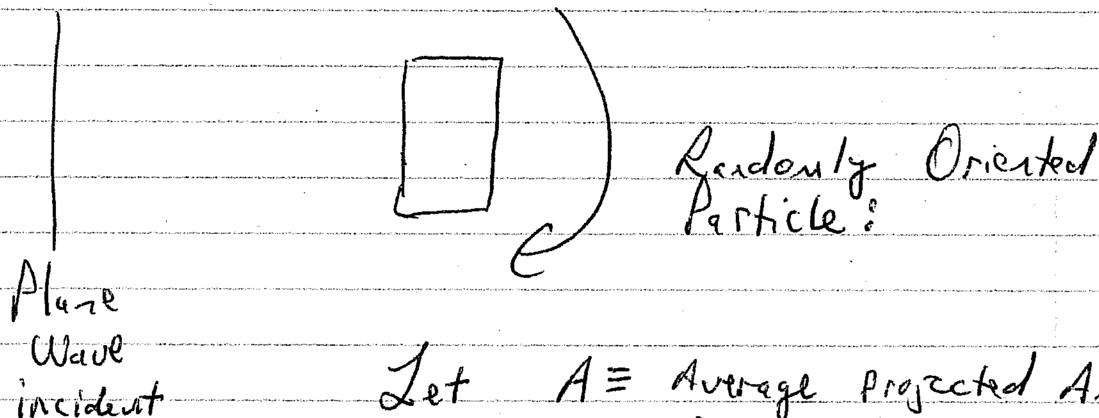
$$\sigma_{abs} = A\left(1 - e^{-\frac{4\pi n_i d}{\lambda}}\right)$$

$$\epsilon = 1 - t$$

Zero Scattering Approximation. Valid when $n_r = 1$.

This simple model will be Useful for class, 26OCT06.

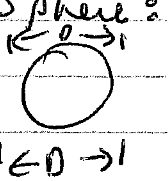
Gross Approximations for σ_{ext}, σ_{abs} for a randomly oriented, generic Particle



Let $A \equiv$ Average projected Area.
$V \equiv$ Particle volume

$$d_{eq} \equiv \frac{V}{A} \quad \text{Equivalent Dimension.}$$

Example Sphere:



$$d_{eq} = \frac{4/3\, \pi D^3/8}{\pi D^2/4} = \frac{2}{3} D$$

Then $\sigma_{ext}(\lambda, d_{eq}) = 2A \left\{ 1 - \cos\left((n_r - 1) \frac{2\pi d_{eq}}{\lambda} \right) \times \right.$

or $\left. e^{-n_i \frac{2\pi d_{eq}}{\lambda}} \right\}$

$\sigma_{ext}(\lambda, d_{eq}) = A\, Q_{ext}(\lambda, d_{eq})$, where

Extinction Efficiency $\equiv Q_{ext} = 2 \{ \}$

Similarly,

$$\sigma_{abs} = A \; Q_{abs}(\lambda, d_{eq})$$

Absorption Efficiency, Unitless.

$$Q_{abs}(\lambda, d_{eq}) = \left(1 - e^{-\frac{4\pi n_i}{\lambda} d_{eq}}\right).$$

This is 'Anomalous Diffraction Approximation' invented by H.C. Van de Hulst.

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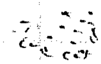


Gas: τ_{ext} = Optical Depth

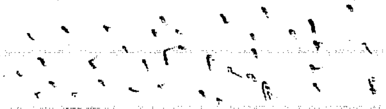
$$= k_{ext} \; \rho_a \; L$$

$$\frac{m^2}{kg} \quad \frac{kg}{m^3} \quad m$$

Dimensionless

Cloud of Particles: Needed for multiple scattering, etc.

$|\longleftarrow L \longrightarrow|$



$$\tau_{ext} = \beta_{ext} \, L$$

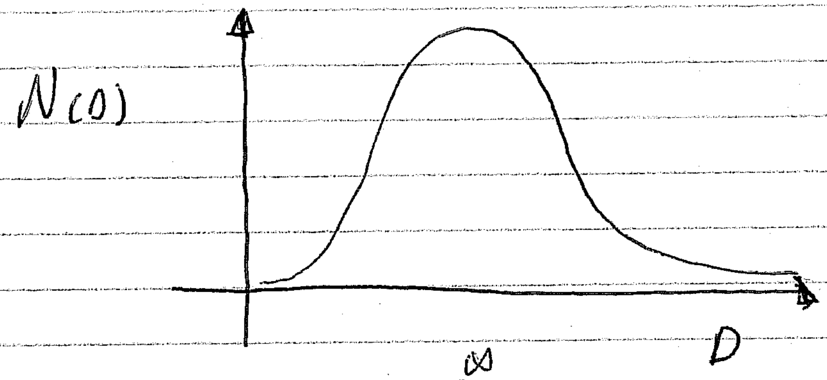
$$\frac{1}{m} \quad m$$

Here $\beta_{ext} = \int_0^\infty N(D) \, \sigma_{ext}(\lambda, D) \, dD$

$$m^2 \qquad m$$

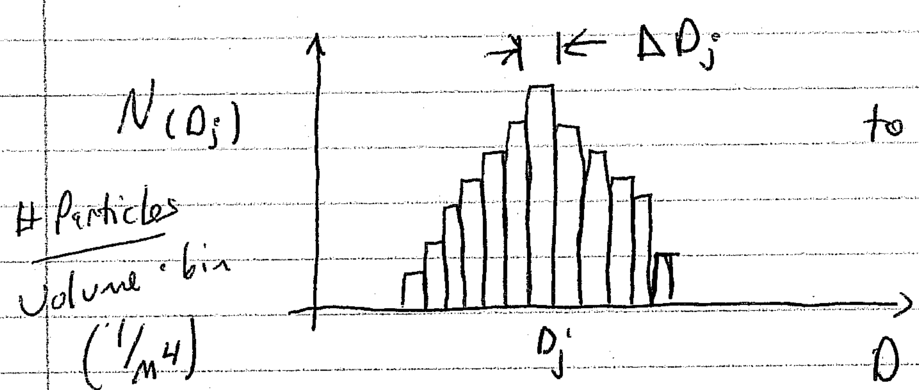
$$\frac{\#\text{ Particles}}{\text{Volume} \cdot \text{Size interval}} \qquad \frac{1}{m^4}$$

Here $N(D)$ is particle size distribution:



$$N_0 = \int_0^\infty N(D)\,dD \Rightarrow \frac{\text{Total \# Particles of all size}}{\text{Volume}}$$

Discrete Problem:



Put particles in bins according to their size.

$$\boxed{\tau_{ext}(\lambda_i) \approx L \sum_{j=1}^{\infty} N(D_j)\,\sigma_{ext}(\lambda_i, D_j)\,\Delta D_j}$$

Similar Arrangement for $\tau_{abs}(\lambda_i)$.

Web Example

Particle Size Spectrum Inversion:

Obtain $N(D_j)$ using spectral extinction measurements $\tau_{ext}(\lambda_i)$ and theoretical $\sigma_{ext}(\lambda_i, D_j)$. Fredholm Integral Equation of 1st Kind – ILL Posed Problem – $N(D_j)$ obtained by inversion is very sensitive to slight errors in measurement and theory.

Matrix Formulation

$$\begin{bmatrix} \tau_{ext}(\lambda_1) \\ (\lambda_2) \\ (\lambda_3) \\ \vdots \\ (\lambda_M) \end{bmatrix} = \begin{bmatrix} \sigma_{ext}(\lambda_1, D_1) & (\lambda_1, D_2) & \cdots \\ & & \\ & & \\ & & \sigma_{ext}(\lambda_M, D_N) \end{bmatrix} \begin{bmatrix} L\, N(D_1)\,\Delta D_1 \\ \vdots \\ L\, N(D_N)\,\Delta D_N \end{bmatrix}$$

We have algorithms to get $N(D_j)$ ...

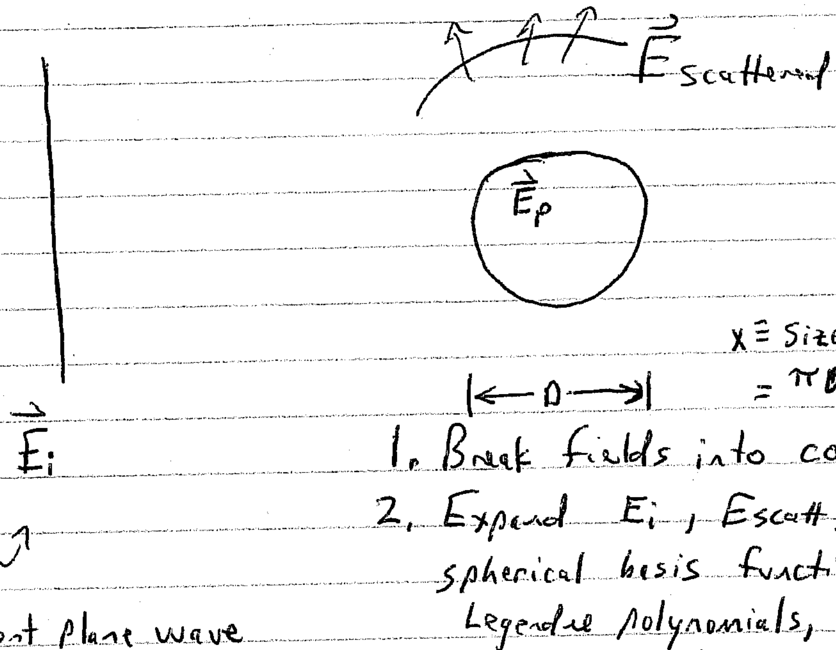
(Works well for spheres)

Lorenz - Mie Scattering Theory

(Mie, circa 1908, optics of gold colloidal suspensions)
Practice
Lorenz - Theory Development, 1800's

It is ... Series solution for Maxwell's Equations of Electrodynamics for the electric and magnetic fields scattered by a homogeneous, isotropic sphere with a plane wave incident.

Nothing more, Nothing less!



$x \equiv$ Size Parameter $= \pi D/\lambda$

$\boxed{\text{Linear Electromagnetic Scattering}}$

1. Break fields into components.
2. Expand E_i, E_{scatt}, E_p in spherical basis functions, Legendre polynomials, spherical Bessel and Neumann functions.
3. Evaluate boundary condition for $\vec{E}$ and $\vec{H}$ on particle boundary.

Only full solution humans can compute for most situations

Gnarly mathematical solution, Contains rainbows, coronae, glories, polarization properties, σ_{ext}, σ_{abs}, σ_{scatt}, g (asymmetry parameter), phase functions, etc, for spheres.