Problem 1

Midterm   My crack @ it.

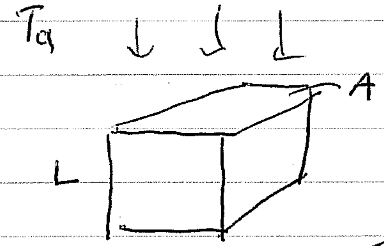
1 A. $\frac{\text{Total \# molecules}}{\text{volume}} = \frac{P}{k_B T} = \frac{N}{V}$. (IGL)

$$\boxed{\frac{\text{Total \# } CO_2}{\text{Volume}} = n_c = \frac{N}{V}\frac{C}{10^6} = \frac{P}{k_B T}\frac{C}{10^6}.}$$

B. Each CO_2 molecule gets a volume $\frac{V}{n}$. Let $a^3 \equiv \frac{\text{Volume}}{\text{molecule}}$ so

$$\boxed{a = \left(1/n_c\right)^{1/3} = 100\left(\frac{k_B T}{PC}\right)^{1/3}}$$

C. Solve first for photons coming from the top.

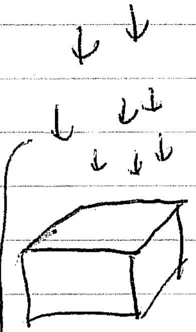


Photon Flux $= \frac{\text{\# Photons}}{\text{Area} \cdot \text{Sec}} = \frac{\text{\# Photons}}{\text{Volume}} \times C$

Photon speed

$LA = V$

The photon flux is given by $\mathfrak{I}$.



Photons / volume travel @ C speed

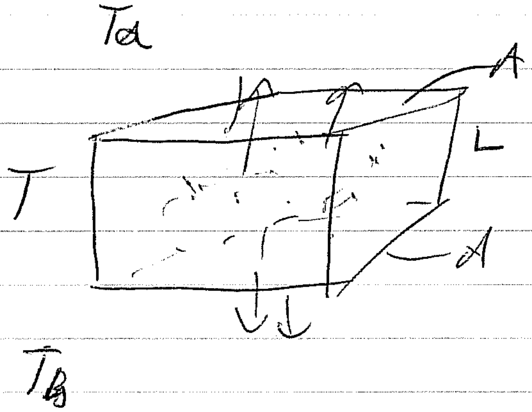
$$\mathfrak{I} = \frac{F\Delta\nu}{hc\nu}, \quad F(T_a) = \underbrace{\left(\frac{\text{\# Photons}}{\text{sec } A}\right)}_{\mathfrak{I}\uparrow}\left(\frac{\text{Energy}}{\text{Photon}}\right)\frac{1}{\text{wavenumber}}$$

Let $n_p \equiv \frac{\text{\# Photons}}{\text{Volume}}$. Then from photon flux above,

$$\boxed{n_p = \frac{\left[F(T_A) + F(T_B)\right]\Delta\nu}{hc^2\nu}}$$

The distance between photons is n_p^{-3}.

D. By working with irradiance, I imply 1-D. Photons go ↑ and ↓ in 1-D.

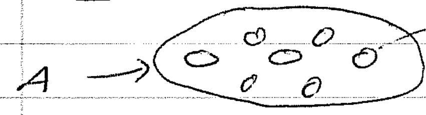


Photons go out the top and bottom (And Sides).

$$\underbrace{\frac{\#\text{ Photons Created}}{\text{Sec}}}_{} = \underbrace{2}_{\uparrow + \downarrow}\,\underbrace{\sigma_A \overbrace{n_c}^{\#CO_2/\text{Volume}} L}_{\text{Emissivity}}\;\underbrace{\frac{F(T)\,\Delta\nu\,A}{hc\nu}}_{\frac{\#\text{ photons}}{\text{sec}}}$$

$\frac{\#\text{ photons}}{\text{sec}}$ through A top and bottom, because of $\leftarrow 2$.

Note $\sigma_A n_c L$ =



σ_A Abs Cross Section / molecule

$n_c L = \frac{\#\text{ molecules}}{\text{Area}}$

When $\sigma_A n_c L \ll 1$, is ok to use this for emissivity. If $\sigma_A n_c L \sim 1$ or greater, then $\epsilon = 1 - e^{-\sigma_A n_c L}$. Why?

Because when $\sigma_A n_c L \ll 1$, it is not likely that photons will see more than 1 σ_A when going from 0 to L. A photon that encounters a σ_A is destroyed, so the form $\epsilon = 1 - e^{-\sigma_a n_c L}$ assumes both random orientation and location of absorbing molecules, and takes care of the fact that at larger $\sigma_a n_c L$ photons destroyed coming in a particular direction, by 1 molecule, can't be destroyed again a little later by another σ_a.

See Bohren, pg 60-66.

Let's see: $\tilde{\epsilon} = \sigma_a n_c L$ or $\epsilon = 1 - e^{-\sigma_a n_c L}$,

if $\tilde{\epsilon} = \epsilon$, no problem.

D. Summary: The exponential form of emissivity takes into account that some created photons in volume LA are absorbed before they can leave. The $\frac{\#\text{ created}}{\text{sec}}$ is

(1)
$$\boxed{\frac{\#\text{ Photons Created}}{\text{Sec}} = 2\sigma_A n_c L \frac{F(T)\,\Delta\nu A}{hc\nu}}.$$

The total # leaving through A at 0 and L is

(2)
$$\frac{\#\text{ Photons Created That leave}}{\text{Sec}} = 2\left(1-e^{-\sigma_A n_c L}\right)\frac{F(T)\,\Delta\nu A}{hc\nu}.$$

Therefore the number of photons created that are absorbed before they can leave the volume is

(3)
$$\frac{\#\text{ Photons Created in } V \text{ and absorbed in } V}{\text{Sec}} = 2\left[\sigma_A n_c L - \left(1-e^{-\sigma_A n_c L}\right)\right]\frac{F(T)\,\Delta\nu A}{hc\nu}$$

E. If $\underbrace{\beta_{sca}}_{\sigma_a n_c} L$ is τ_{abs}, $T = e^{-\tau_{abs}}$ = transmission, $Abs = (1-T)$ = Absorption.

Abs is the probability a photon is absorbed in L path. For small τ_{abs}, $e^{-\tau_{abs}} \approx 1 - \tau_{abs}$, so Abs is

$$Abs \approx \sigma_a n_c L.$$

Photons destroyed per Second, coming from outside, =

(4)
$$\left[F(T_a) + F(T_b)\right]\underbrace{\left(\sigma_a n_c L\right)}_{\left(1-e^{-\sigma_a n_c L}\right) \text{ for } \sigma_a n_c L \text{ not} \ll 1}\frac{\Delta\nu A}{hc\nu},$$

The total # of photons destroyed are the sum of the general forms of

Eq(3) and Eq(4)

Photons created within V, but absorbed before they can sneak out.

outside photons absorbed

Photon Destruction in Total = $\frac{\text{Photon Destruction in Total}}{\text{sec}}$, in V,

$$(5)\quad 2\left[\sigma_a n_c L - \left(1-e^{-\sigma_a n_c L}\right)\right]\frac{F(T)\,\Delta\nu A}{hc\nu} +$$

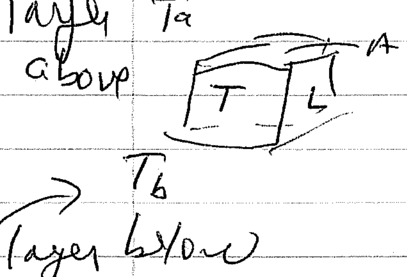
$$\left[F(T_a)+F(T_b)\right]\left(1-e^{-\sigma_a n_c L}\right)\frac{\Delta\nu A}{hc\nu}$$

If, by design we have Thermodynamic Equilibrium between V and the layer above and below, as indicated by $T_a = T_b = T$, then,

from Eq(5), Photon destruction in total =

$$(6)\quad \boxed{2\sigma_a n_c L\,\frac{F(T)\,\Delta\nu A}{hc\nu} = \text{Photon destruction when } T_a = T_b = T}$$

Note: when thermodynamic equilibrium rules, $T_a = T_b = T$



then (Eq1) = (Eq6) destruction and creation balance.

Radiative Equilibrium can happen when $2F(T) = F(T_a) + F(T_b)$, external sources needed

In summary, we see that Kirchoff's law that $\epsilon = abs$ is mirrored in photon destruction = photon creation when equilibrium reigns.

Numerics

F A $9.18 \times 10^{21} \frac{CO_2 \text{ Molecules}}{m^3} = 9.18 \times 10^{15} \frac{CO_2}{cm^3}$

G B 47.8 nm between CO_2 molecules

H C $2.89 \times 10^{10} \frac{\text{photons}}{m^3}$, 326 μm between photons

I D $10^{19} \frac{\text{Photons created}}{\text{sec}}$

J E $10^{19} \frac{\text{Photons absorbed}}{\text{sec}}$

Rigorously for our case, T same all around

\# Photons per second into A, top and bottom is

$$\frac{2F(T)\Delta\nu A}{hc\nu} \frac{\text{Photons}}{\text{sec}} = 8.66 \times 10^{18} \frac{\text{photons}}{\text{sec}}$$

This exceeds the # photons created ~~at~~ and absorbed,

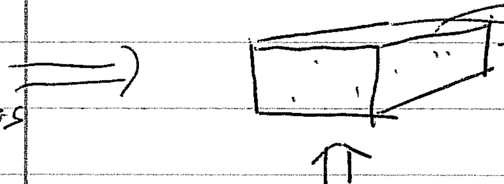
because some created in the volume are actually absorbed before leaving it.

Note: $\tau_{abs} = \sigma_a n_c L = 1.175$ is > 1, is not small, so it is important to look at the whole story, carefully. Interesting what you find when you make up a problem and solve it in detail later.

Problem 2 Summary: Midterm

Aside: Consider a small volume.

Light absorbs inside



A = area
L = height
m = mass
ρ = density

C_p = heat Capacity / mass

Q = heat in from lamp turned on for τ seconds. ΔT = resulting temperature change.

Let Q be due to radiation absorbed through whole volume.

$$Q = \underbrace{\alpha\left(\tfrac{m^2}{g}\right)}_{\frac{\text{Absorption cross section}}{\text{mass}}} \underbrace{\tau\ F_0}_{\text{Energy in light}} \overbrace{A \ \underbrace{\rho\ L}_{\frac{\text{mass}}{\text{area}}}}^{\text{mass}}$$

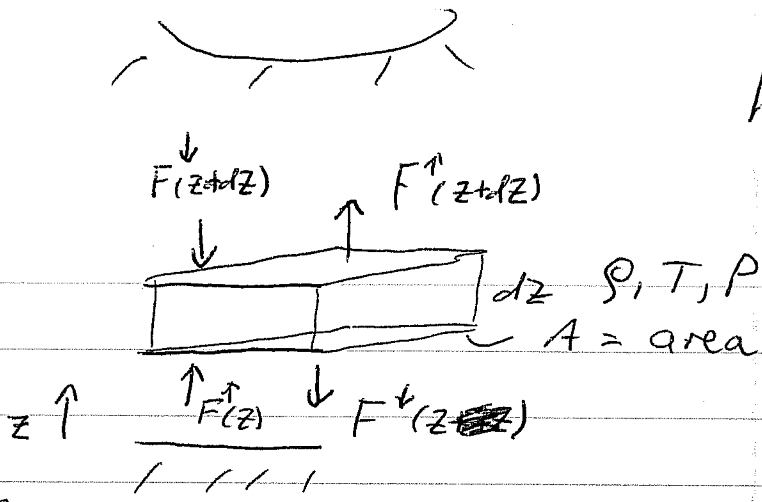
$$(sec)\ (W/m^2)\ (m^2)\ (g/m^3)(m)$$

$$= m\, C_p\, \Delta T \ , \quad m = \rho A L$$

So $$\boxed{\Delta T = \frac{\alpha F_0 \tau}{C_p}}$$

This is independent of mass since both the heat capacity and the amount of radiation absorbed depend on mass.

C_p large, low ΔT $\Rightarrow$ Energy gets distribution to many molecular modes - rotation, vibration, translation. Large α, large ΔT, much radiation absorbed.



Prob 2

A.

$F^{\uparrow}(z) = \int F^{\uparrow}(\lambda, z)\, d\lambda \Rightarrow$ total irradiance $\uparrow$ at z. Etc for $\downarrow$.

Net irradiance $\equiv F^{\uparrow}(z) - F^{\downarrow}(z) = F^{net}(z)$

Accounting Problem:

(1) irradiance $(In - Out) A =$ Absorbed Optical Power $= \rho_0 c_p \frac{dT}{dt} A dz$

Here C_p heat capacity/mass, $(\rho_0 A dZ)$ mass, $\partial T/\partial t$ rate of temperature change

Over time, the temperature increases. absorption [The layer responds ⇒ motion]

(2) $$(In - out) = F^{\uparrow}(z) + F^{\downarrow}(z+dz) - F^{\downarrow}(z) - F^{\uparrow}(z+dZ)$$
$$= \left(F^{\uparrow}(z) - F^{\downarrow}(z)\right) - \left(F^{\uparrow}(z+dZ) - F^{\downarrow}(z+dZ)\right)$$
$$= F^{net}(z) - F^{net}(z+dZ)$$
$$= -dF^{net}(z) \quad \text{for small } dZ.$$

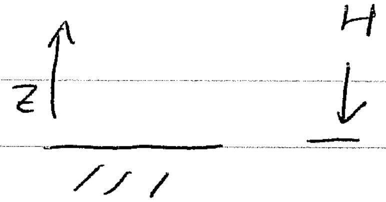
Combining (1) & (2), A divides out,

(3) $$\boxed{\rho C_p \frac{\partial T}{\partial t} = -\frac{dF^{net}(z)}{dZ}}$$

(|) Sunlight

$F_0(\lambda) = $ T.O.A Sunlight

Prob 2 B.



How much sunlight is left at z?

$$F(\lambda, z) = F_0(\lambda)\, e^{-\tau_{abs}(\lambda, z)} \quad \text{where}$$

$$\tau_{abs}(z) = \int_z^H \beta_{abs}(\lambda, z')\, dz'.$$ [Note careful use of dummy z']

Note: Sign convention on Net

For now, let's take this as the net irradiance at z. $F^{net}(z) = -\int_0^\infty F(\lambda, z)\, d\lambda$. Then to get a heating rate,

(4) $$\frac{dF^{net}}{dz} = -\int_0^\infty F_0(\lambda)\left(\frac{-d\tau_{abs}}{dz}\right) e^{-\tau_{abs}(\lambda, z)}\, d\lambda$$

Then

(5) $$\frac{d\tau_{abs}(z)}{dz} = -\beta_{abs}(z)$$ {Fundamental thm calculus}

Now (5) → (4) → (3) gives

(6) $$\boxed{\frac{\partial T}{\partial t} = \frac{1}{\rho C_p} \int_0^\infty \beta_{abs}(\lambda, z)\, F_0(\lambda)\, e^{-\tau_{abs}(\lambda, z)}\, d\lambda}$$

Comment: Sign convention on F^{Net} must be viewed seriously. Biggest assumption @ this point is

$$F(\lambda, z) = F_0(\lambda)\, e^{-\tau_{abs}(\lambda, z)}$$

Leaves out scattering, Must λ integrate to get all photons.

(7) Prob 2 C. For dry air, $C_p = 7/2\, R$, $\rho = P/RT$,

$$\rho C_p = 7/2\; P/T.$$

Remark: $C_p = C_v + R$, $C_v = \underbrace{3/2\, R}_{\text{translational Energy}} + \underbrace{2/2\, R}_{\text{rotational Energy}}$

Assuming $1/\lambda$, $\beta_{abs}(\lambda) = \beta_{abs}(\lambda_0)\, \lambda_0/\lambda$.

Now

Grossly assume sunlight at surface is ≈ that @ T.O.A:

Sun Radius — Sun as black body.

(8)
$$F_o(\lambda)\, e^{-\tau_{abs}(\lambda,z)} \approx \left(\frac{r_s}{r_{se}}\right)^2 \frac{2\pi h c^2}{\lambda^5\left(e^{hc/\lambda K T_{sun}}-1\right)} \cos\theta_t$$

Sun earth distance. Solar Zenith angle

(8) → (7) → (6)

(9)
$$\frac{\partial T}{\partial t} = \frac{2}{7}\frac{T}{P}\, \beta_{abs}(\lambda_0)\, \lambda_0 \left(\frac{r_s}{r_{se}}\right)^2 2\pi h c^2 \cos\theta_t \int_0^\infty \frac{d\lambda}{\lambda^6\left(e^{hc/\lambda K T}-1\right)}$$

Aside: Solve the integral.

Let $x = hc/\lambda k T$ $\quad \lambda = hc/xKT$ $\quad dx = \frac{-hc}{\lambda^2 K T_s}\, d\lambda = -\frac{K T_s x^2}{hc}\, d\lambda$

(10)
$$I = \left(\frac{K T_{sun}}{hc}\right)^5 \left[\int_0^\infty \frac{x^4\, dx}{e^x - 1} \Rightarrow 1.037 \times 4! = 24.89\right]$$

(10) & (9) give

(11)
$$\boxed{\frac{\partial T}{\partial t} = (1+A)\left(\frac{14.2\pi T}{P}\right) \lambda_0\, \beta_{abs}(\lambda_0)\, \frac{(k T_{sun})^5}{h^4 c^3} \left(\frac{r_s}{r_{se}}\right)^2 \cos\theta_t}$$

$$\frac{\;\frac{(k_B T_{sun})^5}{h^4 c^3}\;}{P} \sim \frac{\text{a radiation pressure rate}}{P} \propto 1/\text{time}$$

D. For the sun as a blackbody, and the Sun-Earth distance

$$F_0 = \frac{2\pi^5}{15}\frac{(k_B T_s)^4}{h^3 c^2}\left(r_s/r_{se}\right)^2 \quad \text{or}$$

$$F_0 = \sigma T_{sun}^4 \left(r_s/r_{se}\right)^2$$

Some of Eq 11 can be simplified, as

$$\boxed{\frac{\partial T}{\partial t} = (1+A)\,\frac{106.5}{\pi^4}\,\frac{T}{p}\,\lambda_0 \beta_{abs}(\lambda_0, z)\,\frac{k T_{sun}}{hc}\,F_0 \cos\theta_t}$$

This is the heating rate of a layer due to absorption with a $1/\lambda$ spectral dependence. Surface albedo A is put in as an adhoc fix for a 2 way path, $\downarrow F_0$ $\uparrow F_0(A)$ so 2 chances for absorption. The product $\lambda_0 \beta_{abs}(\lambda_0)$ is a measure of absorption strength. The term kT_{sun} is reflected in the λ^{-1} dependence of absorption.

F. Exaple from Mexico city

Parameter	MKS Value	Comment
A	0.2	from recent workshop,
T	300K	
P	7×10^{4} Pa	773 mb – Mexico City
λ_0	532×10^{-9} m	532 nm
β_{abs}	$100\ \text{Mm}^{-1} = 10^{-4}\ \text{m}^{-1}$	
T_{sun}	5760 K	
F_0	1380 W/m^2	annual average
h	6.63×10^{-34} J.s	
c	3×10^{8} m/s	
θ_t	30°	$\cos\theta_t = \sqrt{3}/2$
86,400	Seconds / day	
k_B	1.38×10^{-23} J/K	

$$\left(\frac{K}{day}\right)\quad \frac{\partial T}{\partial t} = 1.2\ \frac{106.5}{\pi^4}\ \frac{300}{7\times10^{4}}\ 532\times10^{-9}\ 10^{-4}\ \frac{1.38\times10^{-23}\ 5760}{6.63\times10^{-34}\ 3\times10^{8}}\ 1380\ \frac{\sqrt{3}}{2} * 86{,}400$$

$$\boxed{\frac{\partial T}{\partial t} = 12.3\ \frac{K}{day}}$$

This is a lot.

F. Criticize:

Left Out: in order

1. Rayleigh scattering by molecules as attenuation source, and source of diffuse radiation. Especially needed for UV.
2. Absorption by gases that reduces solar flux @ the surface – O_3, H_2O, SO_2, NO_2, etc.
3. Better solution – go back to Eq (6), replace

$$F_0(\lambda)\, e^{-\tau_{abs}(\lambda, z)} \Rightarrow \overset{\text{measured}}{F(\lambda)}$$

solar flux @ z

where $\beta_{abs}(z)$ is also measured.

4) Surface albedo is function of λ also.

Fix:

$F^{\downarrow}(\lambda)$ $F^{\uparrow}(\lambda)$

$\boxed{M1}$ $\boxed{M2}$ $\boxed{M3}$ $\beta_{abs}(\lambda)$ } all @ z

Measure downwelling solar flux as function λ

Measure upwelling solar flux

measure light absorption

$$\frac{\partial T}{\partial t} = \frac{1}{\rho c_p} \int_0^{\infty} \underbrace{\left(F^{\downarrow}(\lambda) + F^{\uparrow}(\lambda)\right) \beta_{abs}(\lambda)}_{\text{all are measurements}}\, d\lambda$$

Bonus:

From Jill Pocock.

$$F^* = \underbrace{F_s^{\downarrow} - F_s^{\uparrow}}_{\text{Solar}} + \underbrace{F_L^{\downarrow} - F_L^{\uparrow}}_{\text{IR}}$$ Net Radiation absorbed by Surface

Surface Response:

$$F^* = F_{HS} + F_{ES} + F_{GS}$$

F_{HS}: Sensible heat flux (feels warm)

F_{ES}: Latent heat flux (from evaporation of water vapor) (later condensation & Latent heat Release)

F_{GS}: Conduction of heat into ground or water

Sensible Heat flux $F_{HS} \approx C_H |V| (T_s - T_{air})$

C_H: Bulk heat transfer coefficient

$|V|$: windspeed

$(T_s - T_{air})$: Surface to air temp diff

Must consider not only root absorption, but IR emission by layer as well, as a cooling effect, and absorption by other means.

The Earth/Atmosphere does this for us every day.