CHAPTER 11

The Radiative Transfer Equation With Scattering

Throughout this book so far, we have discussed scattering primarily as one of two mechanisms for *extinguishing* radiation, the other mechanism being absorption. Thus, the extinction coefficient β_e could be decomposed into the sum of the absorption coefficient β_a and scattering coefficient β_s. The single scatter albedo $\tilde{\omega}$, defined as β_s/β_e, was introduced as a convenient parameter describing the relative importance of absorption and scattering: when $\tilde{\omega} = 0$, extinction of radiation is entirely by way of absorption; when $\tilde{\omega} = 1$, then there is no absorption, only scattering.

When radiation is extinguished via scattering, its energy is not converted to another form; rather, the radiation is merely redirected. The *loss* of radiation along one line-of-sight due to scattering is therefore always associated with a *gain* in radiation along other lines-of-sight passing through the same volume.

You can easily observe the above phenomenon with the help of a powerful, narrow beam of light, such as that from a searchlight, an automobile headlight or a laser pointer. When the air is very clear — i.e., free of smoke, dust, haze, or fog — then the beam can pass directly in front of you and you will not see it, because essentially none of the radiation is scattered out of the original path into the direction toward your eyes. But if the air contains suspended particles, then these particles will scatter some fraction of the beam into all directions, including toward your eyes, and the path of the beam will be clearly apparent, especially against a dark background. In this case, scattering clearly serves as a *source* of radiation, as seen from your vantage point. Of course the original beam is depleted by the same process. For example, if a fog is thick enough, the headlights of an oncoming car can't be seen at all until it is relatively close to you.

This chapter introduces the terminology and mathematical notation required to account for scattering as a *source* of radiation in the radiative transfer equation.

11.1 When Does Scattering Matter?

When scattering is important as a source of radiation along a particular line-of-sight, then the complexity of calculations of radiative transfer along that line-of-sight greatly increases compared with the nonscattering case. This is because one must, in the worst case, solve for the intensity field not just in *one direction* along a *one-dimensional path* but for *all directions* simultaneously in *three-dimensional space*! You would therefore like to be able to neglect scattering (as a source, at least) whenever you can get away with it.

In fact, you can safely ignore scattering as a source whenever gains in intensity due to scattering along a line-of-sight are negligible compared with (a) losses due to extinction *and* (b) gains due to thermal emission. In the atmosphere, these conditions are *usually* satisfied for radiation in the thermal IR band and for microwave radiation when no precipitation (e.g., rain, snow, etc.) is present. In addition, if one is concerned only with the depletion of *direct* radiation from an isolated, point-like source, such as the sun, then the above conditions are usually satisfied to reasonable accuracy.

For virtually any problem involving the interaction of shortwave (ultraviolet, visible, and near-IR) radiation with the atmosphere, scattering is the dominant atmospheric source of radiation along any line-of-sight other than that looking directly at the sun. The blue sky, white or gray clouds, the atmospheric haze that reduces the visual contrast of distant objects — all of these make their presence known primarily by way of scattered radiation.

11.2 Radiative Transfer Equation with Scattering

11.2.1 Differential Form

Previously, we derived Schwarzschild's Equation (8.4) under the assumption that scattering was unimportant and that therefore $\beta_e = \beta_a$. Under that assumption, we found that the change in intensity dI along an infinitesimal path ds could be written as

$$dI = dI_{\text{abs}} + dI_{\text{emit}} , \tag{11.1}$$

where the depletion due to absorption is given by

$$dI_{\text{abs}} = -\beta_a I \, ds , \tag{11.2}$$

and the source due to emission is

$$dI_{\text{emit}} = \beta_a B(T) \, ds . \tag{11.3}$$

In order to generalize the equation to include scattering, we must recognize that depletion occurs due to both absorption and scattering, so that β_e rather than β_a must appear in the depletion term. Moreover, we must now add a source term that describes the contribution of radiation scattered *into* the beam from other directions, so that

$$dI = dI_{\text{ext}} + dI_{\text{emit}} + dI_{\text{scat}} , \tag{11.4}$$

where

$$dI_{\text{ext}} = -\beta_e I \, ds . \tag{11.5}$$

The term dI_{scat} requires more thought. First, we know it must be proportional to the scattering coefficient β_s, since without scattering there can be no contribution from this term. Second, we recognize that radiation passing through our infinitesimal volume from *any* direction $\hat{\Omega}'$ can potentially contribute scattered radiation in the direction of interest $\hat{\Omega}$. Moreover, these contributions from all directions will sum in a linear fashion — that is, the path taken by a photon arriving from one direction is not influenced by the presence of, or paths taken by, other photons.

Mathematically, these ideas are expressed as follows:

$$dI_{\text{scat}} = \frac{\beta_s}{4\pi}\left[\int_{4\pi} p(\hat{\Omega}', \hat{\Omega}) I(\hat{\Omega}') \, d\omega'\right] ds , \tag{11.6}$$

where the integral is over all 4π steradians of solid angle, and the scattering *phase function* $p(\hat{\Omega}', \hat{\Omega})$ is required to satisfy the normalization condition

$$\boxed{\frac{1}{4\pi}\int_{4\pi} p(\hat{\Omega}', \hat{\Omega})\, d\omega' = 1\,.} \quad (11.7)$$

The complete differential form of the radiative transfer equation can thus be written

$$dI = -\beta_{\mathrm{e}} I ds + \beta_{\mathrm{a}} B ds + \frac{\beta_{\mathrm{s}}}{4\pi}\int_{4\pi} p(\hat{\Omega}', \hat{\Omega}) I(\hat{\Omega}')\, d\omega'\, ds\,. \quad (11.8)$$

Dividing through by $d\tau = -\beta_{\mathrm{e}} ds$, we can write

$$\boxed{\frac{dI(\hat{\Omega})}{d\tau} = I(\hat{\Omega}) - (1-\tilde{\omega})B - \frac{\tilde{\omega}}{4\pi}\int_{4\pi} p(\hat{\Omega}', \hat{\Omega}) I(\hat{\Omega}')\, d\omega'\,,} \quad (11.9)$$

where, in the interest of clarity, we make the dependence of I on direction $\hat{\Omega}$ explicit.

This is the most general and complete form of the radiative transfer equation that we will normally have to deal with in this book.[1]

Note that it is often convenient to lump all sources of radiation into a single term, so that (11.9) may be written in shorthand form as

$$\boxed{\frac{dI(\hat{\Omega})}{d\tau} = I(\hat{\Omega}) - J(\hat{\Omega})\,,} \quad (11.10)$$

where the *source function* is given by

$$\boxed{J(\hat{\Omega}) = (1-\tilde{\omega})B + \frac{\tilde{\omega}}{4\pi}\int_{4\pi} p(\hat{\Omega}', \hat{\Omega}) I(\hat{\Omega}')\, d\omega'\,.} \quad (11.11)$$

[1]We have defined $d\tau$ here to be negative for translation toward the detector. This means that negative terms on the right hand side are source terms, and positive terms imply depletion. Some textbooks use the opposite convention, in which case the signs of all terms on the right hand side are reversed.

We see that the total source is a weighted sum of thermal emission and scattering from other directions, with the single scatter albedo controlling the weight given to each. If $\tilde{\omega} = 0$, the scattering term vanishes; if $\tilde{\omega} = 1$, the thermal emission component vanishes.

11.2.2 Polarized Scattering†

Throughout most of this book, we have ignored the role of polarization in atmospheric radiative transfer and considered the effects of transmission, absorption and scattering only on the *scalar* intensity I. Although this is almost always an approximation, it is often a very good one. There are times, however, when it is necessary to revert to a more accurate *fully polarized* treatment of radiative transfer, which requires us to consider changes not only in I but in all elements of the four-parameter Stokes vector $\mathbf{I} = (I, Q, U, V)$ that was introduced in (2.52). The fully polarized version of the differential radiative transfer equation (11.9) can be written

$$\frac{d\mathbf{I}(\hat{\Omega})}{d\tau} = \mathbf{I}(\hat{\Omega}) - (1 - \tilde{\omega})B\mathbf{U} - \frac{\tilde{\omega}}{4\pi}\int_{4\pi} \mathbf{P}(\hat{\Omega}', \hat{\Omega})\mathbf{I}(\hat{\Omega}')\, d\omega' \,, \quad (11.12)$$

where $\mathbf{P}(\hat{\Omega}', \hat{\Omega})$ is a 4×4 scattering *phase matrix*, and $\mathbf{U} \equiv (1, 0, 0, 0)$ when $\tilde{\omega}$ is considered to be independent of polarization. The latter assumption is not guaranteed to be valid; indeed, for some problems involving preferentially oriented particles, such as might be encountered in ice clouds or snowfall, even $\tilde{\omega}$ and the extinction coefficient β_e (implicit in τ) may each depend on both polarization and direction.

You are most likely to encounter the fully polarized RTE in the context of certain remote sensing problems. A more comprehensive discussion of polarized radiative transfer (though still with some simplifications, such as polarization-independent extinction and optical path) is given by L02 (Section 6.6). For the remainder of this book, we will continue to rely on the scalar form of the RTE given by (11.9) unless otherwise noted.

11.2.3 Plane Parallel Atmosphere

Although we know that the atmosphere is far from horizontally homogeneous, especially where clouds are concerned, most analytic solutions and approximations to the radiative transfer equation with scattering have been derived for the plane parallel case. Why? There are three basic reasons:

- Plane-parallel geometry is really the only semi-realistic case that lends itself to straightforward analysis and/or numerical solution (e.g., in climate and weather forecast models).
- There are indeed problems (e.g., the cloud-free atmosphere, horizontally extensive and homogeneous stratiform cloud sheets) for which the plane-parallel assumption usually seems quite reasonable as an approximation to reality.
- Even where it is not reasonable, there remains considerable doubt about the best way(s) to handle three-dimensional inhomogeneity, especially when computational efficiency is essential. Therefore investigators tend to fall back on plane-parallel geometry (with minor embellishments, such as the so-called *independent pixel approximation*), knowing that it is not perfect but believing it to be better than nothing at all (this is fine, as long as the potential for large errors is understood by all concerned!).

To adapt (11.10) to a plane-parallel atmosphere, we reintroduce the optical depth τ, measured from the top of the atmosphere, as our vertical coordinate, and we will henceforth use $\mu \equiv \cos\theta$ to specify the direction of propagation of the radiation measured from zenith.[2] We then have

$$\mu \frac{dI(\mu,\phi)}{d\tau} = I(\mu,\phi) - J(\mu,\phi)\,, \tag{11.13}$$

where the source function for both emission and scattering is

$$J(\mu,\phi) = (1-\tilde{\omega})B + \frac{\tilde{\omega}}{4\pi}\int_0^{2\pi}\int_{-1}^{1} p(\mu,\phi;\mu',\phi')I(\mu',\phi')\,d\mu' d\phi'\,. \tag{11.14}$$

[2]Some textbooks, such as L02 and S94, specify that $\mu \equiv \cos\theta$. Others, such as TS02, instead define $\mu \equiv |\cos\theta|$, as I also did in an earlier chapter of this book. When writing the equations of radiative transfer with scattering, each convention has its own advantages and disadvantages. Here I have chosen the definition that permits the same equation to be used for both upward and downward radiation.

There is only a relatively small class of applications in which it is necessary to consider both scattering and emission at the same time. Two examples include (1) microwave remote sensing of precipitation, and (2) remote sensing of clouds near 4 μm wavelength, for which scattered solar radiation may be of comparable importance to thermal emission. Except where noted, the rest of this book will focus on problems involving scattering of solar radiation only, without the additional minor complication of thermal emission.

11.3 The Scattering Phase Function

One way to give physical meaning to the scattering phase function is to regard $\frac{1}{4\pi}p(\hat{\Omega}', \hat{\Omega})$ as a probability density: Given that a photon arrives from direction $\hat{\Omega}'$ and is scattered, what is the probability that its new direction falls within an infinitesimal element $d\omega$ of solid angle centered on direction $\hat{\Omega}$? The normalization condition (11.7) simply ensures that energy is conserved when there is no absorption ($\tilde{\omega} = 1$); i.e., the new direction of a scattered photon is guaranteed to fall *somewhere* within the available 4π steradians of solid angle, and you can't get more (or fewer) photons out than you put in.

The functional dependence of the phase function on $\hat{\Omega}$ and $\hat{\Omega}'$ can be quite complicated, depending on the sizes and shapes of the particles responsible for the scattering. Nevertheless, an important simplification can be made when particles suspended in the atmosphere are either spherical or else randomly oriented. For example, cloud droplets are spherical, and small aerosol particles and air molecules, while generally not spherical, have no preferred orientation.[3] In such cases, the scattering phase function for a volume of air depends only on the angle Θ between the original direction $\hat{\Omega}$ and the scattered direction $\hat{\Omega}'$, where

$$\boxed{\cos\Theta \equiv \hat{\Omega}' \cdot \hat{\Omega}\,.} \tag{11.15}$$

[3]Falling ice crystals, snowflakes, and raindrops generally *do* have a preferred orientation due to aerodynamic forces, and this directional anisotropy must sometimes be considered in radiative transfer calculations.

The ability to replace $p(\hat{\Omega}', \hat{\Omega})$ with $p(\hat{\Omega}' \cdot \hat{\Omega}) \equiv p(\cos\Theta)$ is very helpful, inasmuch as the number of independent directional variables needed to fully characterize p is reduced from four (two each for $\hat{\Omega}$ and $\hat{\Omega}'$) to only one. The normalization condition (11.7) then reduces to

$$\frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} p(\cos\Theta)\ \sin\Theta d\Theta d\phi = 1\,, \tag{11.16}$$

or

$$\boxed{\frac{1}{2} \int_{-1}^{1} p(\cos\Theta)\ d\cos\Theta = 1\,.} \tag{11.17}$$

Except where noted, this simplified notation for the phase function will be utilized throughout the remainder of this book.[4]

11.3.1 Isotropic Scattering

The simplest possible scattering phase function is one that is constant; i.e.

$$p(\cos\Theta) = 1\,. \tag{11.18}$$

Scattering under this condition is known as *isotropic.* It describes the case that all directions $\hat{\Omega}$ are equally likely for a photon that has just been scattered. Thus, the new direction the photon takes is in no way predictable from the direction it was traveling prior to being scattered; in other words, the photon "forgets" everything about its past.

An example of the random path of a single photon experiencing isotropic scattering is shown in Fig. 11.1a. Note that once the photon passes into the interior of the cloud layer it wanders aimlessly, often changing directions quite sharply with each scattering. Eventually, its "drunkard's walk" takes it back to the cloud top, where it emerges and, in this case, contributes to the albedo of the cloud. A different random turn at any point in its path could have instead taken it to the cloud base, where it would have then contributed

[4]It sometimes necessary, however, to recast a phase function that is inherently of the form $p(\cos\Theta)$ in terms of the absolute directions $(\hat{\Omega}', \hat{\Omega})$ in order to facilitate integration over zenith and/or azimuth angles θ and ϕ.

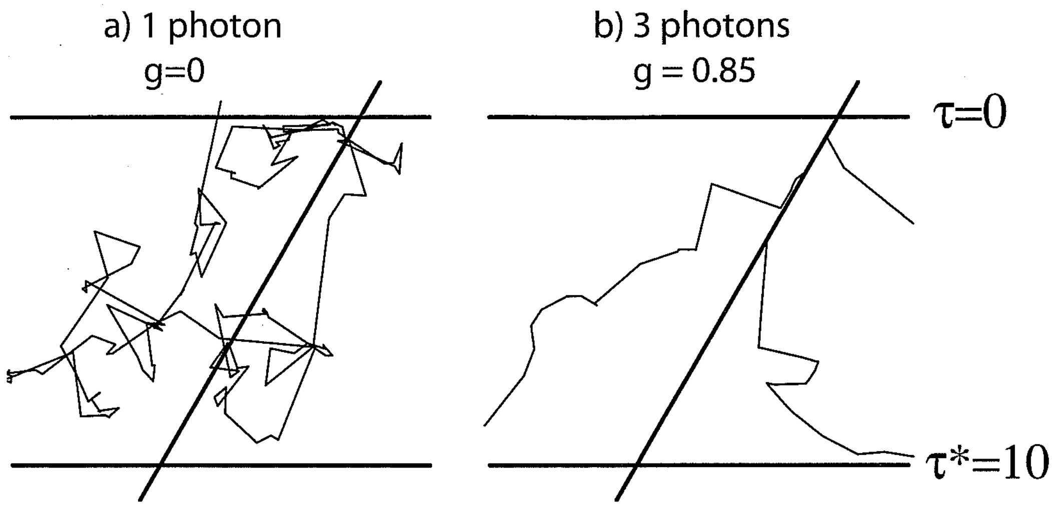

Fig. 11.1: Examples of the random paths of photons in a plane-parallel scattering layer with optical thickness $\tau^* = 10$. Photons are incident from above with $\theta = 30°$. Heavy diagonal lines indicate the path an unscattered photon would take. (a) The trajectory of a single photon when scattering is isotropic. (b) The trajectories of three photons when asymmetry parameter $g = 0.85$, which is typical for clouds in the solar band.

to the diffuse transmittance. Note that because of the large optical depth, the *direct* transmittance of the layer is vanishingly small; therefore there is virtually no chance that the photon could have passed all the way through to cloud base without first being scattered numerous times.

For isotropic scattering, the scattering source term in the radiative transfer equation simplifies to

$$\frac{\tilde{\omega}}{4\pi}\int_{4\pi} p(\hat{\Omega}', \hat{\Omega}) I(\hat{\Omega}')\, d\omega' \quad \rightarrow \quad \frac{\tilde{\omega}}{4\pi}\int_{4\pi} I(\hat{\Omega}')\, d\omega' \; . \quad (11.19)$$

That is, the source is independent of both $\hat{\Omega}$ and $\hat{\Omega}'$ and is simply equal to the single scatter albedo times the spherically averaged intensity.

Scattering by real particles in the atmosphere is never even approximately isotropic. Nevertheless, because the assumption of isotropic scattering leads to important simplifications in the analytic solution of the radiative transfer equation, it is frequently employed in theoretical studies in order to gain at least qualitative insight into the behavior of radiation in a scattering medium.

Furthermore, for some kinds of radiative transfer calculations, it is possible to find approximate solutions to a problem involving nonisotropic scattering by recasting it as an equivalent isotropic scattering problem, for which analytic solutions are easily obtained. Such so-called *similarity transformations* will be discussed in a later chapter.

11.3.2 The Asymmetry Parameter

In order compute scattered intensities to a high degree of accuracy, it is necessary to specify the functional form of the phase function $p(\cos\Theta)$. As will be seen in Chapter 12, the phase functions of real atmospheric particles can be complex and don't lend themselves to simple mathematical descriptions. Often, however, we don't care about intensities at all but only fluxes. In such cases, it is not necessary to get bogged down with details of the phase function; rather, it is sufficient to know the relative proportion of photons that are scattered in the forward versus backward directions. The scattering *asymmetry parameter* g contains this information and is defined as

$$\boxed{g \equiv \frac{1}{4\pi} \int_{4\pi} p(\cos\Theta) \cos\Theta \, d\omega \,.} \tag{11.20}$$

The asymmetry parameter may be interpreted as the average value of $\cos\Theta$ for a large number of scattered photons. Thus

$$-1 \leq g \leq 1 \,. \tag{11.21}$$

If $g > 0$, photons are preferentially scattered into the forward hemisphere (relative to the original direction of travel), while $g < 0$ implies preferential scattering into the backward hemisphere. If $g = 1$, this is the same as scattering into exactly the same direction as the photon was already traveling, in which case it might as well not have been scattered at all! A value of -1, on the other hand, implies an exact reversal of direction with every scattering event, a special case that is imaginable but physically unlikely.

For isotropic scattering, as discussed in the previous subsection, we expect $g = 0$, since scattering into the forward and backward hemispheres is equally likely. This can be shown explicitly by substituting $p = 1$ into (11.20), expanding $d\omega$ in spherical polar coordinates as $\sin\theta d\theta d\phi$, and choosing $\hat{\Omega} = \hat{z}$ so that the scattering angle Θ is the same as the zenith angle θ. Thus,

$$\begin{aligned} g &= \frac{1}{4\pi}\int_0^{2\pi}\int_{-\pi/2}^{\pi/2} \cos\theta \sin\theta \; d\theta \; d\phi \\ &= \frac{1}{2}\int_{-\pi/2}^{\pi/2} \cos\theta \sin\theta \; d\theta \\ &= \frac{1}{2}\int_{-1}^{1} \mu \; d\mu \\ &= 0 \,. \end{aligned} \tag{11.22}$$

Note that while $g = 0$ for isotropic scattering, other phase functions can also have $g = 0$ and not be isotropic. The best example is the Rayleigh phase function derived in section 12.2, which describes the scattering of radiation by particles much smaller than the wavelength.

For many problems of interest, such as scattering of solar radiation in clouds, the asymmetry parameter g falls in the range 0.8–0.9. In other words, cloud droplets are strongly *forward scattering* at solar wavelengths. Fig. 11.1b shows examples of photon paths for $g = 0.85$. Although the average distance traveled by a photon between scattering events is the same as for isotropic scattering (Fig. 11.1a), the photon is now far more likely to be scattered into a direction that is not too different from its previous direction of travel. As a result, the photon's path, while still random, is far less chaotic than the isotropic case. Statistically, the photon travels a much greater distance before experiencing a sharp reversal in course. It is therefore also more likely to reach the cloud base and less likely to exit at cloud top. In other words, we expect the diffuse transmittance to increase and the cloud-top albedo to decrease when the asymmetry is large.

11.3.3 The Henyey-Greenstein Phase Function

The scattering phase functions of particles are often rather complicated (we will return to this subject in Chapter 12). As already

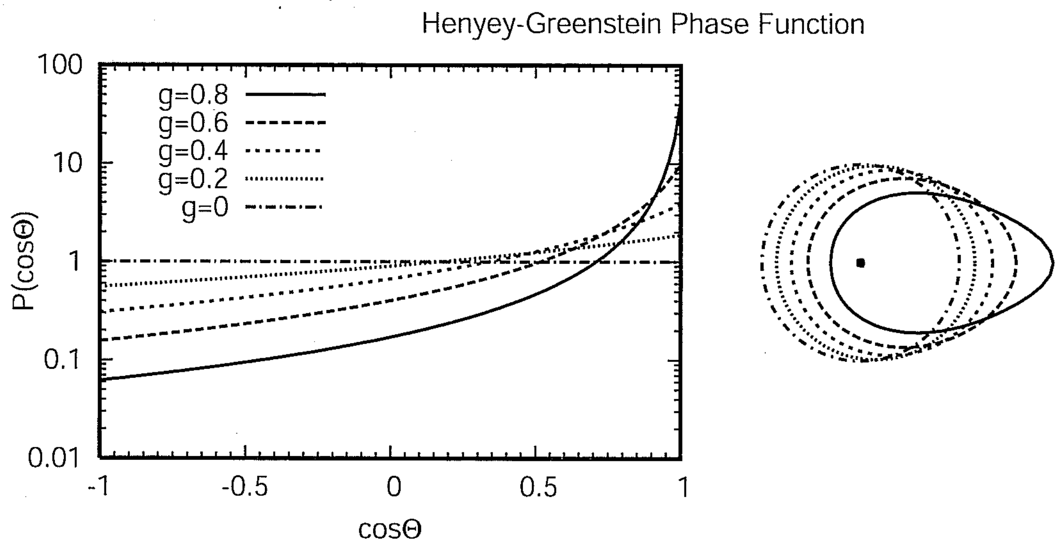

Fig. 11.2: The Henyey-Greenstein phase function plotted versus cos(Θ) (left) and as a log-scaled polar plot (right).

pointed out, it is not always necessary to use a complete and accurate description of $p(\cos\Theta)$ in a radiative transfer calculation, as long as we know the asymmetry parameter g. For some types of calculations, we might want to employ a "stand-in" phase function that satisfies the following criteria:

- It should have a convenient mathematical form, ideally one that is an explicit function of the desired asymmetry parameter g.
- It should bear at least some resemblance to the shape of real phase functions, even if it doesn't have details like the rainbow, corona, etc. (See Chapter 12.)
- In order to be physically meaningful, the value of the phase function should be nonnegative for all values of Θ.

The *Henyey-Greenstein* phase function is the most widely used "model" phase function that satisfies all of the above criteria. It is given by

$$p_{HG}(\cos\Theta) = \frac{1-g^2}{(1+g^2-2g\cos\Theta)^{3/2}} \, . \qquad (11.23)$$

As you can see from Fig. 11.2, the HG phase function is isotropic for $g = 0$. For positive g, the function peaks increasingly in the forward direction but remains quite smooth. In other words, it captures the asymmetry of real phase functions rather well but not the higher order details.

Problem 11.1: Show that the parameter g appearing in (11.23) equals the asymmetry parameter as defined by (11.20).

Although the HG phase function with $g > 0$ does a good job of reproducing the observed forward peak in the phase functions of real particles, there is often also a pronounced (but somewhat smaller) backward peak which is *not* captured. Therefore, you will sometimes see the use of a *double* HG function, with one of the two terms serving to represent the backward peak:

$$p_{HG2}(\cos\Theta) = b p_{HG}(\cos\Theta; g_1) + (1-b) p_{HG}(\cos\Theta; g_2) , \quad (11.24)$$

where $g_1 > 0$, $g_2 < 0$, and $0 < b < 1$.

Problem 11.2: (a) Given g_1, g_2, and b, find the asymmetry parameter g of the double Henyey-Greenstein phase function.

(b) For marine haze particles in the visible band, it has been found that good values for the above parameters are $b = 0.9724$, $g_1 = 0.824$, and $g_2 = -0.55$. Find g.

(c) Plot the phase function $p(\cos\Theta)$ described in part (b), using a logarithmic vertical axis.

11.4 Single vs. Multiple Scattering

When a solar photon enters the atmosphere or a cloud layer from the top, it will eventually either exit again (top or bottom) or else get absorbed. There are no other possibilities. Before either one happens, however, the photon may experience anywhere from zero to a very large number of scatterings from atmospheric particles.

Recall that if the photon passes entirely through the layer without getting either scattered or absorbed, then it is said to be *directly*

$t_{dir} = e^{-\tau_{sca}}$ $\tau^* = \tau_{sca}$

transmitted. The probability of this happening to any particular photon is given by the direct transmittance t_{dir}, which we already know how to compute from Beer's Law. If, on the other hand, the photon exits the layer after having been scattered at least once, then it contributes to either the diffuse transmittance or the albedo of the layer, depending on whether it exits at the bottom or the top, respectively.

It is helpful now to distinguish between two general classes of problems: those in which *single scattering* dominates and those in which *multiple scattering* is the rule. In the first case, almost all of the photons contributing to the albedo and/or diffuse transmittance were scattered exactly once. Single scattering prevails whenever the layer is optically thin — i.e., $\tau^* \ll 1$, because each photon that is scattered in the interior of the layer then has a high probability of exiting the cloud before getting scattered a second time. Single scattering is also favored whenever the layer is strongly absorbing ($\tilde{\omega} \ll 1$), since a photon is then much more likely to get absorbed than to get scattered a second time.

If, on the other hand, the layer is both optically thick ($\tau^* > 1$) *and* strongly scattering ($1 - \tilde{\omega} \ll 1$), then many or most of the photons that enter the layer will be scattered more than once, perhaps even hundreds of times, before reemerging at the base or top of the layer. It takes a fair amount of sophistication to solve multiple scattering problems accurately. In fact, whole textbooks have been devoted to just this subject. In this book, we will defer until Chapter 13 our own fairly rudimentary treatment of radiative transfer with multiple scattering.

RTE for Single Scattering

For now, let's focus on the much simpler single scattering problem. In the absence of thermal emission, (11.13) and (11.14) can be combined to give

$$\mu \frac{dI(\mu,\phi)}{d\tau} = I(\mu,\phi) - \frac{\tilde{\omega}}{4\pi} \int_0^{2\pi} \int_{-1}^{1} p(\mu,\phi;\mu',\phi') I(\mu',\phi')\, d\mu' d\phi' . \tag{11.25}$$

What makes the single scattering problem simple is that the intensity $I(\mu', \phi')$ inside the integral is then, by definition, the attenuated intensity from the direct source (e.g., the sun) with no significant contribution from radiation that has already been scattered. If we assume a parallel beam of incident radiation from a point source above the cloud (a good approximation for direct sunlight), then we can write

$$I(\mu', \phi') = F_0 \delta(\mu' - \mu_0) \delta(\phi' - \phi_0) e^{\frac{\tau}{\mu_0}} \,, \tag{11.26}$$

where $\mu_0 < 0$ and ϕ_0 give the directions of the incident beam, F_0 is the solar flux normal to the beam, and the exponential term is just the direct transmittance from the layer top $\tau = 0$ to level τ within the cloud. The Dirac δ-function $\delta(x)$ is defined to be zero for all $x \neq 0$ and infinite for $x = 0$, and it is normalized so that $\int_{-\infty}^{\infty} \delta(x') \, dx' = 1$, and $\int_{-\infty}^{\infty} f(x') \delta(x' - x) \, dx' = f(x)$.

With the above substitutions, (11.25) reduces to

$$\mu \frac{dI}{d\tau} = I - \frac{F_0 \tilde{\omega}}{4\pi} p(\cos\Theta) e^{\tau/\mu_0} \,, \tag{11.27}$$

where the dependence of I on μ, ϕ, and τ is understood, and where $\cos\Theta \equiv \hat{\Omega} \cdot \hat{\Omega}_0$ is the cosine of the angle between the incident sunlight and the direction of the scattered radiation.

Let's rearrange the above equation and multiply through by $e^{-\tau/\mu}$:

$$\frac{dI}{d\tau} e^{-\tau/\mu} - \frac{1}{\mu} I e^{-\tau/\mu} = -\frac{F_0 \tilde{\omega}}{4\pi\mu} p(\cos\Theta) e^{\tau/\mu_0} e^{-\tau/\mu} \,. \tag{11.28}$$

This allows us to rewrite the left hand side as a simple derivative of a single expression:

$$\frac{d}{d\tau} \left[I e^{-\tau/\mu} \right] = -\frac{F_0 \tilde{\omega}}{4\pi\mu} p(\cos\Theta) e^{\tau\left(\frac{1}{\mu_0} - \frac{1}{\mu}\right)} \,. \tag{11.29}$$

In order to compute the scattered intensity emerging from the top or bottom of the atmosphere, we just have to integrate the above equation from $\tau = 0$ to $\tau = \tau^*$. For simplicity, we will assume here that $\tilde{\omega}$ and the phase function are independent of height, so that we can take them outside the integral. We get

$$I(\tau^*) e^{-\frac{\tau^*}{\mu}} - I(0) = \frac{-F_0 \tilde{\omega}}{4\pi\mu \left(\frac{1}{\mu_0} - \frac{1}{\mu} \right)} p(\cos\Theta) \left[e^{\tau^* \left(\frac{1}{\mu_0} - \frac{1}{\mu} \right)} - 1 \right] \,. \tag{11.30}$$

It may surprise you to learn that the above equation is valid both for downwelling radiation at the bottom of the atmosphere and for upwelling radiation at the top of the atmosphere. In the first case, we're interested in $I(0)$ for the case that $\mu > 0$:

$$I(0) = I(\tau^*)e^{-\frac{\tau^*}{\mu}} + \frac{F_0\tilde{\omega}}{4\pi\mu\left(\frac{1}{\mu_0} - \frac{1}{\mu}\right)} p(\cos\Theta) \left[e^{\tau^*\left(\frac{1}{\mu_0} - \frac{1}{\mu}\right)} - 1\right] . \tag{11.31}$$

In the second case, we want $I(\tau^*)$ for $\mu < 0$, which requires only a slight rearrangement:

$$I(\tau^*) = I(0)e^{\frac{\tau^*}{\mu}} - \frac{F_0\tilde{\omega}}{4\pi\mu\left(\frac{1}{\mu_0} - \frac{1}{\mu}\right)} p(\cos\Theta) \left[e^{\frac{\tau^*}{\mu_0}} - e^{\frac{\tau^*}{\mu}}\right] . \tag{11.32}$$

To summarize, the above equations give scattered radiances at the top and bottom of the atmosphere (or a thin cloud layer) for the special case that all of the above are satisfied: (a) multiple scattering is negligible, (b) $\tilde{\omega}$ and $p(\cos\Theta)$ are constant, and (c) the sole external illumination is a parallel beam source such as the sun. *It's important to recall the requirement that either $\tilde{\omega} \ll 1$ and/or $\tau^* \ll 1$ in order for the first of these requirements to be satisfied.*

Let's take things a step further. First, we'll focus only on the scattered atmospheric contribution to the radiance and drop the term that describes the direct transmission of radiation from the opposite side of the atmosphere (we can always add it back, if we want it). Second, we'll assume that the reason why we can neglect multiple scattering is that $\tau^* \ll 1$, and we'll further assume that μ_0 and μ are *not* much smaller than one. Taking advantage of the fact that, for small x, $e^x \approx 1 + x$, we can then simplify our equations to

$$\left.\begin{array}{ll} \text{For } \mu > 0, & I(0) \\ \text{For } \mu < 0, & I(\tau^*) \end{array}\right\} = \frac{F_0\tilde{\omega}\tau^*}{4\pi|\mu|} p(\cos\Theta) . \tag{11.33}$$

The interpretation of the above equation is straightforward — so straightforward in fact, that we probably could have guessed it without going through all the previous steps. First, the quantity $F_0\tau^*$ tells us the magnitude of the extinguished solar flux (recall that this is valid only in the limit of small τ^*). Second, the quantity $(\tilde{\omega}/4\pi)p(\cos\Theta)$ tells us how much of the intercepted flux contributes to the scattering source term in a given new direction $\hat{\Omega}$. Finally, the factor $1/\mu$ accounts (to first order) for the fact that you are looking through less atmosphere if you view it vertically than if you look toward the horizon; consequently the path-integrated contribution of scattering to the observed intensity increases toward the horizon.

Problem 11.3: If you have access to a decent plotting program, set things up so that you can conveniently plot $I(\tau^*)$ versus $-1 < \mu < 0$ using both (11.33) and (11.32) on the same graph. Assume no extraterrestrial source of radiation from direction $\mu \neq \mu_0$. Assume isotropic scattering. Determine the range of μ, μ_0, and τ^* for which the second equation is a good approximation to the first. When the two disagree significantly, describe the nature of the disagreement. Focus on values of $\tau^* \leq 0.1$, since we know that neither equation is valid unless the atmosphere is optically thin. Note also that (11.32) cannot be directly evaluated when $\mu_0 = \mu$, though it gives physically reasonable values in the limit as $\mu \to \mu_0$.

11.5 Applications to Meteorology, Climatology, and Remote Sensing

11.5.1 Intensity of Skylight

We imposed several seemingly drastic restrictions in deriving (11.33): $\tau^* \ll 1$, $\tilde{\omega}$ and $p(\cos\Theta)$ independent of τ, μ and μ_0 not too small. In fact, these assumptions are reasonably well justified for molecular scattering of visible and near-IR sunlight in the cloud- and haze-free atmosphere, as long as (a) you stay away from the blue and violet end of the spectrum, and (b) you don't get too close to the horizon.

Therefore, to evaluate the radiant intensity of the sky (apart from the direct rays of the sun itself) you need only specify the optical depth τ^* of the cloud-free atmosphere at the wavelength in question, supply a suitable phase function $p(\Theta)$, and substitute these into (11.33) for arbitrary μ and μ_0.

As will be shown in Chapter 12, the scattering phase function of air molecules in the visible band is

$$p(\Theta) = \frac{3}{4}(1 + \cos^2 \Theta) . \tag{11.34}$$

This so-called *Rayleigh* phase function is quite smooth and is perfectly symmetric with respect to forward and backward scattering ($g = 0$). The factor-of-two variation in intensity implied by the above phase function is relatively minor and is unlikely to be obvious to the eye, especially since it is such a smooth function of the scattering angle Θ. Consequently, we expect the radiant intensity of the sky to appear rather uniform, punctuated only by the narrow spike of high intensity associated with the directly transmitted light of the sun.

Although $p(\Theta)$ has the same shape for molecular scattering at all visible wavelengths, the optical depth τ^* of the cloud free atmosphere is a strong function of wavelength. In fact, it is shown in the next chapter that $\tau^* \propto \lambda^{-4}$. Thus, (11.33) implies that the intensity of skylight due to molecular scattering should also be proportional to λ^{-4} and, indeed, it is precisely this dependence that gives us the blue sky. It is also because τ^* stops being "small" at shorter wavelengths that we can't trust (11.33) to give us accurate sky intensities in the blue and ultraviolet part of the spectrum.

Of course, even the cleanest air found in nature contains not only molecules, but other kinds of particles called aerosols. There are typically many thousands of aerosol particles in every cubic centimeter of air. Those of interest to us here have sizes ranging from 10^{-2} μm to $\sim$1 μm or larger. The scattering of visible light by such comparatively large particles (compared to molecules, that is!) is not as strongly dependent on wavelength as is molecular scattering; furthermore the scattering phase function for aerosols is not symmetric like the Rayleigh phase function but rather exhibits fairly strong forward scattering.

We can summarize the comparative scattering behavior of air molecules and aerosols in the solar band as follows:

	Molecules	**Aerosol**
Wavelength dependence:	λ^{-4}	weak
$p(\Theta)$:	smooth, symmetric	strongly asymmetric
Time/location dependence:	nearly constant	highly variable

Problem 11.4: Based on the above information, explain how the presence of scattering aerosols (e.g., haze) would be expected to visibly affect both (a) the color of the sky and (b) the angular dependence of the intensity of scattered sunlight. Is your analysis consistent with everyday experience?

11.5.2 Horizontal Visibility

Every hour, at tens of thousands of locations around the globe, detailed weather observations are made by trained observers or automated weather instruments. It is no coincidence that a large majority of these stations are associated with airports. It was the need for timely local weather observations in support of aviation, more than any other single factor, that led to the emergence of a dense global weather observing network during the twentieth century.

Although pilots care about a lot of weather variables, the two that are most often of critical concern are (a) cloud ceiling height and (b) horizontal *visibility*. Both affect pilots' ability to safely land at airports and to see and avoid other air traffic. It is the latter variable we will address here, since it is closely tied to the subject of this chapter.

Visibility is defined as the maximum horizontal distance over which the eye can clearly discern features like runways, obstacles, navigation lights, etc. On a clear day in the desert, visibility often exceeds 100 km. But in a pea-soup fog on the California coast, visibility may be measured in meters rather than kilometers.

On first confronting this problem, your initial assumption might be that visibility is controlled entirely by the extinction coefficient β_e along the line-of-sight. After all, the transmittance over a distance s is just

$$t = e^{-\beta_e s} . \tag{11.35}$$

One might argue, therefore, that there is some minimum transmittance t_{min} associated with the limit of human perception, so that the visibility V should be related to β_e as follows:

$$V = -\frac{1}{\beta_e} \log(t_{min}) . \tag{11.36}$$

But such an analysis is too simple. Consider the following examples:

- Translucent ("two-way") mirrors are often used in department stores to facilitate the detection of shoplifters by security personnel. The transmittance is the same for light traveling in either direction through the mirror, but a shopper in a brightly lit room viewing the mirror from the reflective side can't normally see what's on the other side and probably doesn't even realize that it transmits at all. The person (or camera) viewing from the nonreflective side, however, can see through easily, especially if they are situated in a darkened room.

- If you let the windshield on your car get moderately dusty, its transmittance is somewhat reduced, but normally this reduction is fairly minor. In fact, when driving away from the sun during daytime, it may be scarcely noticeable. But if you turn in the direction of the setting sun, you may suddenly find it almost impossible to see! What has changed? Not the transmittance, but rather the glare of light scattered in your direction by the coating of dust particles.

From the above examples, we can perhaps begin to appreciate that it's not *transmittance* but rather *visual contrast* that determines what we can and can't see. We will define the contrast here as the fractional difference between the apparent brightness (radiant intensity) I of an object and the brightness I' of its surroundings:

$$C \equiv \frac{I' - I}{I'} . \tag{11.37}$$

In a purely absorbing atmosphere, a mere reduction in transmittance along a line-of-sight has no impact on the visual contrast between two objects at the same distance, and therefore relatively little on visibility (up to the limit imposed by your eyes' sensitivity to light), as long as the fractional reduction in brightness is the same for both.

Atmospheric scattering reduces contrast by adding a source of radiation to the line-of-sight that is independent of the brightness of whatever is at the far end of the path. Since this source is integrated along the line-of-sight, a long path produces a greater reduction in contrast than a short path. The distance at which the contrast of an object is reduced to the minimum level required for visual detection defines the visibility.

Let's analyze the visibility problem quantitatively, by considering the contribution of single-scattered radiation to the radiance along a finite horizontal path s. Because of the latter condition, we can't use the plane-parallel form of the RTE but must start with an adaptation of (11.9):

$$\frac{dI}{d(\beta_e s)} = -I + J\,, \tag{11.38}$$

where I is the intensity measured horizontally in azimuthal direction ϕ, J is the scattering source function given by

$$J = \frac{\tilde{\omega}}{4\pi}\int_{4\pi} p(\mu_0, \phi_0; 0, \phi) I(\hat{\Omega}')\, d\omega'\,, \tag{11.39}$$

and s is the distance in the direction *toward* the observer.

For this problem, we can assume a horizontally homogeneous atmosphere, so that both the extinction coefficient β_e and the scattering source function J are constant along the line-of-sight. With these assumptions, we can integrate (11.38) to get

$$I(S) = I(0)e^{-\beta_e S} + \left(1 - e^{-\beta_e S}\right) J\,, \tag{11.40}$$

where $I(0)$ is the "intrinsic" radiance of the remote scene as seen without any intervening atmosphere, and $I(S)$ is the brightness of the same scene at the observer's distance S. We see that the observed intensity is just a weighted average of the intrinsic brightness of the distant object and the scattering source function, with the weight being the path transmittance $t = e^{-\beta_e S}$ for the first term and $1-t$ for the second. Obviously, if $t=0$, we see only the atmospheric scattering and no trace of the object at $s=0$.

Problem 11.5: Fill in the steps of the derivation of (11.40) from (11.38). Hint: Multiplying both sides by an integrating factor $e^{\beta_e s}$ will allow you to recast the differential equation into a form that can be directly integrated.

Now let's use the above equation to compute the contrast of a black object with $I(0)=0$ viewed against a white background with intensity $I'(0)$:

$$C = \frac{I'(S)-I(S)}{I'(S)} = \frac{I'(0)t}{I'(0)t+(1-t)J} \,. \tag{11.41}$$

We are interested in the distance S corresponding to the minimum contrast that still permits the human eye to distinguish the object from its background, so we invert the above equation to get

$$S = \frac{1}{\beta_e}\ln\left[\frac{I'(0)(1-C)}{CJ}+1\right] \,. \tag{11.42}$$

Now all that is left is to make reasonable assumptions about $I'(0)$, C, and J.

For the background, we assume an intensity $I'(0)=\alpha F_0$, where α depends on the reflective properties of the background for the particular viewing geometry and direction of the incident sunlight. For example, if the background is a nonabsorbing Lambertian reflector, then $\alpha \leq 1/\pi$, with the equality applying in the case of normal solar incidence.

As before, we'll assume that the atmosphere is optically thin in the vertical and that the sun is high in the sky, so the scattering source function can be approximated as

$$J \approx \frac{F_0\tilde{\omega}}{4\pi}p(\mu_0,\phi_0;0,\phi) \,, \tag{11.43}$$

where μ_0 is the cosine of the solar zenith angle. We'll assume that the phase function can be expressed in terms of the cosine of the scattering angle alone, with

$$\begin{aligned} \cos\Theta &= \hat{\Omega}_0 \cdot \hat{\Omega} \\ &= \left(\sqrt{1-\mu_0^2}\cos\Delta\phi, \sqrt{1-\mu_0^2}\sin\Delta\phi\ , \mu_0\right) \cdot (1,0,0) \\ &= \sqrt{1-\mu_0^2}\cos\Delta\phi \end{aligned} \tag{11.44}$$

where $\Delta\phi = \phi - \phi_0$ is the angle between the viewing azimuth and the solar azimuth.

We can now substitute the above expressions for J and $I'(0)$, with $\alpha \approx \mu_0/\pi$ and $C \approx 0.02$, to get

$$S \approx \frac{1}{\beta_e} \ln\left[\frac{200\mu_0}{\tilde{\omega} p(\cos\Theta)} + 1\right] . \tag{11.45}$$

Problem 11.6: Use (11.45) together with the phase function for marine haze given in Problem 11.2 to plot the visibility in km versus azimuth $\Delta\phi$ relative to the sun's direction for two cases: $\mu_0 = 1$ and $\mu_0 = 0.5$. For both cases assume $\beta_e = 1.0\ \text{km}^{-1}$ and $\tilde{\omega} = 1$. Explain the differences between the two curves. Are your results consistent with your experience?