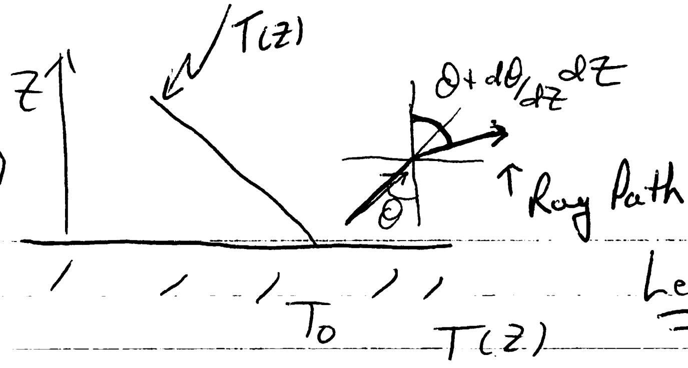
Problem
Calculate Sound Path in atmosphere.



Let $T(z) \equiv T_0 - \Gamma_0 z$
const

$n = \dfrac{c_0}{c(z)} = n(z)$ = refractive index for sound as a function of height.

(1) Snell's law gives $n_1 \sin\theta = \text{const} = \left(n(z) + \dfrac{dn}{dz}dz\right) \times \sin\left(\theta + \dfrac{d\theta}{dz}dz\right)$

Aside: $\sin\left(\theta + \dfrac{d\theta}{dz}dz\right) =$

(2) $$\underbrace{\sin\theta\cos\left(\frac{d\theta}{dz}dz\right)}_{\cos \simeq 1} + \underbrace{\sin\left(\frac{d\theta}{dz}dz\right)}_{\simeq \frac{d\theta}{dz}dz}\cos\theta$$

Using (2) in (1),

$$n\sin\theta = \left(n + \frac{dn}{dz}dz\right)\left(\sin\theta + \frac{d\theta}{dz}dz\cos\theta\right)$$

$$\cancel{n\sin\theta} \simeq \cancel{n\sin\theta} + n\,d\theta\cos\theta + dn\sin\theta$$

(3) so $$\frac{dn}{n} = -\frac{\cos\theta}{\sin\theta}d\theta$$

with $n(z) = \dfrac{c_0}{c(z)} \Rightarrow \dfrac{dn}{n} = -\dfrac{\cancel{c_0}}{c(z)^2}\dfrac{dc}{dT}\dfrac{dT}{dz}dz \cancel{(dz)} \big/ \cancel{c_0}$

$$\frac{dn}{n} = -\frac{1}{c(z)}\frac{dc}{dT}\frac{dT}{dz}dz = +\Gamma_0\frac{1}{c(z)}\frac{dc}{dT}dz$$

Using $c(z) = (\gamma R T)^{1/2}$, $\dfrac{dc}{dT} = \dfrac{(\gamma R)^{1/2}}{2}\dfrac{1}{T^{1/2}}$

$$\frac{dn}{n} = +\frac{1}{(\cancel{\gamma R}T)^{1/2}}\frac{(\cancel{\gamma R})^{1/2}}{2T^{1/2}}\Gamma_0\,dz$$

$$\frac{dn}{n} = \frac{\Gamma_0}{2T}dz$$

Then $$\frac{\Gamma_0}{2}\int_0^z \frac{dz}{T_0 - \Gamma_0 z} = -\int_{\theta_0}^{\theta}\frac{\cos\theta}{\sin\theta}d\theta$$

$u = \sin\theta$, $du = \cos\theta\, d\theta$

$U = T_0 - \Gamma_0 z$, $dU = -\Gamma_0 dz$

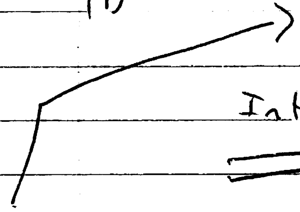
$$= -\frac{1}{2}\int_{T_0}^{T_0 - \Gamma_0 z}\frac{dU}{U} = \int_{\sin\theta}^{\sin\theta_0}\frac{du}{u} = \ln\left(\frac{\sin\theta_0}{\sin\theta}\right)$$

$$\ln\left(\frac{T_0}{T_0 - \Gamma_0 z}\right)^{1/2} = \ln \sin\theta_0/\sin\theta$$

Solving,

$$\sin\theta = \sin\theta_0 \left(\frac{T_0-\Gamma_0 z}{T_0}\right)^{1/2}$$

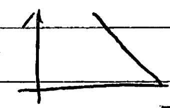
(4)



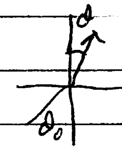
$$\boxed{\sin\theta = \sin\theta_0 \left(1-\frac{\Gamma_0 z}{T_0}\right)^{1/2}}$$

Interpretation;

$\Gamma_0 > 0$, z



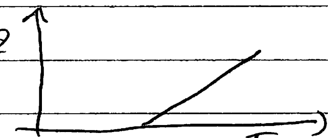
$T(z)$ $\Rightarrow$



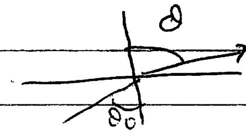
Upward refraction. $\theta_0 > \theta$

Can go immediately to Pg 3 for Ray Path

$\Gamma_0 < 0$ z



$T(z)$



$\theta > \theta_0$, and

$\sin\theta = 1$ when $\sin\theta_0\left(1-\frac{\Gamma_0 z}{T_0}\right)^{1/2} = 1$

Turning Point

$$1-\frac{\Gamma_0 z}{T_0} = \sin^{-2}\theta_0$$

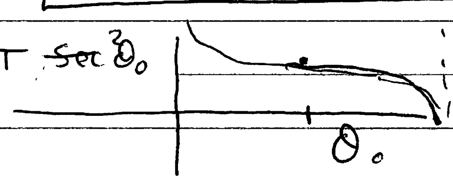
$$z = \frac{T_0}{\Gamma_0}\left(1-\frac{1}{\sin^2\theta_0}\right)$$

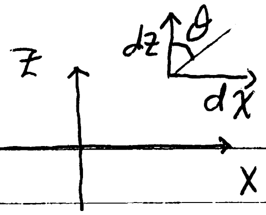
or

$$z = -\frac{T_0}{\Gamma_0}\left(\frac{1-\sin^2\theta_0}{\sin^2\theta_0}\right)$$

$$\boxed{z = -\frac{T_0}{\Gamma_0}\cancel{\sec}\cot^2\theta_0}$$

$-\frac{z\Gamma_0}{T_0}$ $\cot$ ~~sec~~ $^2\theta_0$





Strategy: The relationship between θ and z are given in (4). The ray path in Cartesian Coordinates is expressed by $x = x(z)$.

(5) **Solution:** Trig says $\quad \dfrac{dx}{dz} = \tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{\sin\theta}{(1-\sin^2\theta)^{1/2}}$.

Eq (4) gives $\sin\theta$ as a function of z, so we can look @ Eq (5) as $\dfrac{dx}{dz} = \text{Function}(z, \sin\theta_0)$.

The integral form of Eq (5) is

(6) $$x - x_0 = \int_0^z \frac{\sin\theta_0 (1 - z/z_0)^{1/2}}{[1 - \sin^2\theta_0 (1 - z/z_0)]^{1/2}}\, dz \quad \text{where}$$

x_0 = constant of integration
$x_0 = 0 \Rightarrow$ initial position

$z_0 \equiv T_0/\Gamma_0$ = adiabatic Top of atmosphere when $\Gamma_0 = 9.8$ K/km, or otherwise is just a number.

Note $z_0 > 0 \Rightarrow$ z↑ T↘

$z_0 < 0$ inversion.

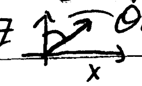
Eq (6) is an integral solution for the ray path $x(z)$. Non dimensionalizing the integral,

$$\text{Let } u = \sin^2\theta_0 (1 - z/z_0)$$

$$dz = -z_0 (du/\sin^2\theta_0).$$

Then (6) becomes nondimensionalized by

$$\frac{\sin^2\theta_0\,(x-x_0)}{z_0} = \int_{\sin^2\theta_0(1-z/z_0)}^{\sin^2\theta_0} \frac{u^{1/2}\,du}{[1-u]^{1/2}}, \qquad (7)$$

where θ_0 is the sound ray launch angle at $z=0$, 

, and

$$z_0 \equiv T_0/\Gamma_0 .$$

Eq. (7) has an interesting integral in it.

For $z \geq -z_0\left(\frac{1}{\sin^2\theta_0} - 1\right)$, $z_0 < 0$ implying inversion,

the denominator of (7) $\to 0$.