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Hyperbolic umbilic diffraction catastrophe and rainbow scattering from spheroidal drops

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The characterization of short-wavelength scattering phenomena has been advanced by the study of diffraction catastrophes[1,2]. Examples include optical phenomena[1–5] and molecular collisions[6]. The most familiar example of a diffraction catastrophe is the ordinary rainbow[7–9]. The angular variation of the scattering from a spherical drop locally corresponds to that of a fold-diffraction catastrophe[1,2], a result most clearly seen with monochromatic illumination. We have studied the scattering from drops whose shape closely approximated that of an oblate spheroid with the short axis—the symmetry axis—vertical. The drops were illuminated by a horizontally propagating gaussian beam with a wavelength $\lambda = 633$ nm. These drops were observed to scatter in the horizontal rainbow region with patterns like those of hyperbolic-umbilic (classification D_4^+) diffraction catastrophes[1,5,6]. Visible D_4^+ diffraction patterns observed previously include light transmitted by frosted glass surfaces[10] and by liquid lenses clinging to tilted glass plates[1,4].

Figure 1 shows the ray diagram of the primary rainbow of a spherical drop. For the once-reflected (twice-refracted) rays there is a minimum scattering angle of $\theta_R \simeq 138°$ and the associated ray is known as the Descartes ray. When the scattering angle θ is between θ_R and 166°, there are two rays of the class shown. The rays labelled 1 and 2 are for $\theta \simeq 152°$. In the two-ray region, intensity oscillations produce the supernumerary arcs[7–9]. Figure 1 is also applicable to rays which lie in the horizontal equatorial plane P of the spheroidal drops; P cuts the drop's surface in a circle and both the incident ray and the local surface normals lie in P.

In our apparatus (see Fig. 2), a horizontal metal plate was coupled to a piezoelectric vibrator which was driven at a frequency of 27 kHz. A curved reflector (not shown in Fig. 2) was placed above the plate's centre so as to establish an acoustic standing wave in the air below the reflector. Drops of distilled water could be levitated in this wave by an upward-directed acoustic radiation pressure. (The levitator is similar, in principle, to one described in ref. 11.) Figure 3 shows a levitated drop. The oblate shape is due to the spatial distribution of the radiation pressure[12]. In addition to the deformation evident in Fig. 3, the drop vibrates at 27 kHz with surface displacements which are too small to affect the light scattering. The drop's diameter will be denoted by D in plane P, and by H along the vertical symmetry axis. Raising the ultrasonic amplitude increases D/H.

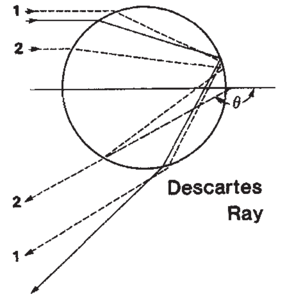

Fig. 1 Rays through either a spherical drop or through a spheroidal drop in the equatorial plane.

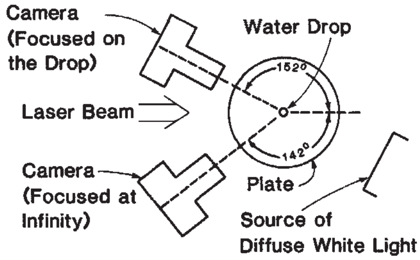

Fig. 2 Schematic view of the apparatus from above. The drop is held 11 mm above the plate and the optical axes of the cameras lie near a horizontal plane containing the drop. The diffuse light is usually turned off.

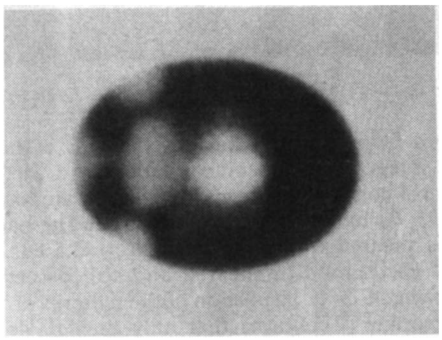

Fig. 3 Photographs of an acoustically levitated drop having a diameter $D = 2.3$ mm. The drop was illuminated both with diffuse light and the laser beam. The four patches on the left are due to refracted laser light.

The scattering was photographed for randomly polarized illumination from a He-Ne laser. The camera was focused on infinity so that the pattern recorded was equivalent to that in the far field (which is at distances $r \gg D^2/\lambda$ from the drop). Horizontal and vertical coordinates in each photograph were linear in the horizontal and vertical scattering angles θ and β, where β is relative to plane P. The camera viewed the region: $136° < \theta < 148°$, $-6° < \beta < 6°$.

Figure 4*a* shows the scattering from a slightly nonspherical drop having $D = 1.9$ mm. The intensity modulations are similar to other photographs of supernumerary arcs with laser illumination[13]. Within experimental uncertainties, their spacing is as predicted by Airy theory[7,9]. Figure 4*b*–*f* are the patterns for another less nearly spherical drop. For Fig. 4*b* the drop had $D = 1.39$ mm and $D/H \simeq 1.23$; smaller values of D/H gave patterns similar to Fig. 4*a*. Changes in the pattern with changes in D/H were also viewed through the camera's lens by eye.

Figure 4*d* manifests the parallelogram and caustic structures characteristic of focal section patterns of D_4^+ diffraction

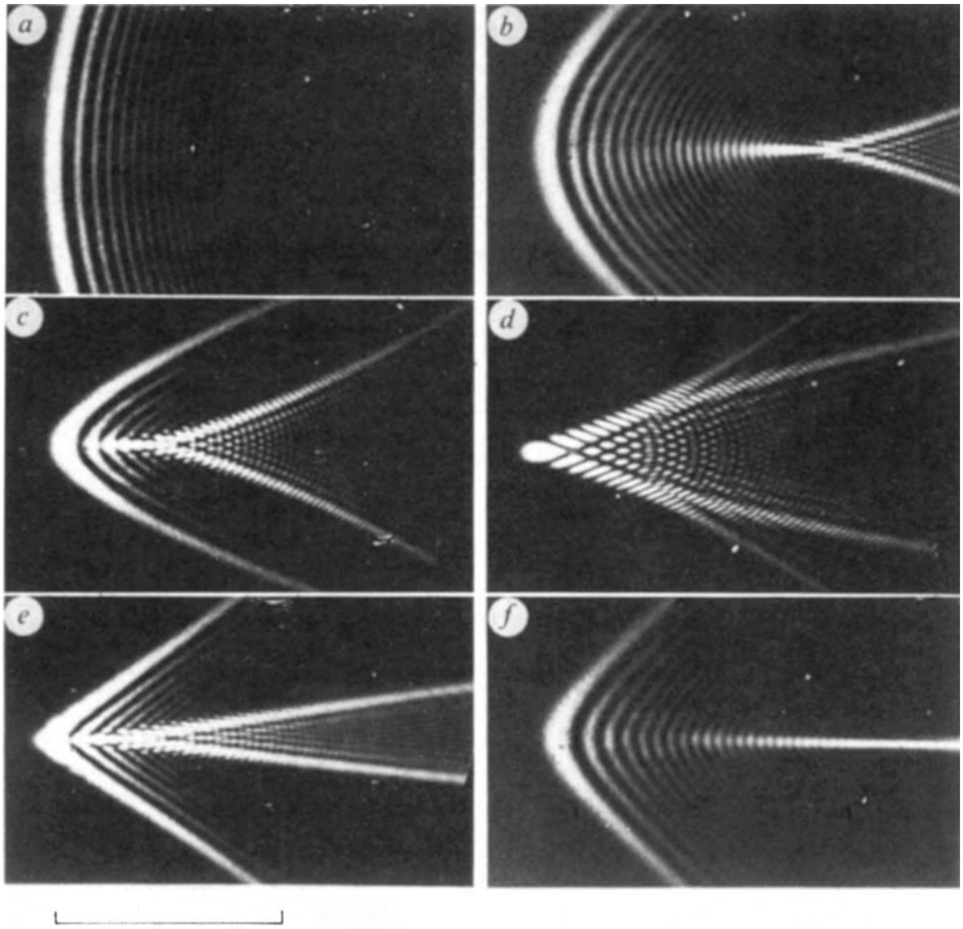

Fig. 4 Photographs of rainbow region scattering patterns arranged in order of increasing drop axis ratio D/H. The horizontal scattering angle θ increases from left to right. The cusp in *b* moved into view from the right as D/H was increased. *d* is the focal section of a hyperbolic-umbilic diffraction catastrophe while *c* and *e* manifest the unfolding of the catastrophe in response to small changes in D/H. Scale bar, 5°.

catastrophes. Consequently, the normal form[1,2] of the associated diffraction integral has the control parameter $C_3 = 0$. Evidently the transverse control parameters C_1 and C_2 are locally related to β and $\theta - \theta_R$ by a linear transformation. The parallelogram structure is sheared with an apex angle $\simeq 43.5 \pm 1°$. Figure 4*b*, *c*, *e* and *f* correspond to sheared and scaled sections of D_4^+ patterns having $C_3 \neq 0$. Inspection of the patterns of the elementary catastrophes[1,4] confirms that only the D_4^+ describes our observations.

When $C_3 \neq 0$, the D_4^+ pattern may be partitioned into three angular regions according to the number of stationary points of the phase function within the diffraction integral[1,5,6]. The number of points corresponds to the number of once-reflected (twice refracted) rays which contribute to the scattering. This partitioning is shown in Fig. 5*a* for the pattern in Fig. 4*c*. That the right-most region has four rays was confirmed by direct observation. A camera having a small aperture, positioned as shown in Fig. 2, was focused on the drop. Figure 5*b* is drawn from a representative image so as to show the locations of the four observed rays. Rays 1 and 2 lie in plane P and correspond to those in Fig. 1. Rays 3 and 4 are horizontally directed, having $\beta \simeq 0°$ when outside the drop. They are skew rays because inside the drop they do not lie in a horizontal plane. These rays were not anticipated from previous studies[13,14] of rainbows of spheroidal drops, which considered only rays confined to a plane.

Other features observed were consistent with those expected of a D_4^+ diffraction catastrophe at short wavelengths $\lambda \ll D$. The axis ratio D/H which gives $C_3 = 0$ should be independent of D for spheroidal drops having the same refractive index. To check this prediction, drops were illuminated with diffuse white light (see Fig. 2) and their profiles were photographed. Before each photograph, D/H was adjusted so that the laser scattering pattern was a focal section as in Fig. 4*d*. Photographs of six drops having D of 1.5–2.4 mm gave axis ratios not correlated with D. The average measured D/H of 1.305 ± 0.016 is a lower limit of the true axis ratio because any misalignment of the camera's optical axis from the equatorial plane would give a systematic reduction in the measured D/H.

For scattering from large spherical drops, Airy's theory[1,7] predicts that when $\theta = \theta_R$, the intensity I at a distance r has $I \propto (D/\lambda)^{1/3}(D/r)^2$. Examination of the D_4^+ diffraction integral[1,5,6] gives $I \propto (D/\lambda)^{2/3}(D/r)^2$ when C_3, β and $\theta - \theta_R$ all vanish. Visual inspections and photographs confirm that the scattering near the apex of the focal section appears brighter than that near θ_R of the other sections such as Fig. 4*b* and *c*. This focusing for $C_3 \simeq 0$ may be observable in nature when the Sun is close to the horizon. Raindrops having $D \simeq 4$ mm have an axis ratio $\simeq 1.3$. However, such large drops are flattened below the centre[15] and the effect on the focusing is not known.

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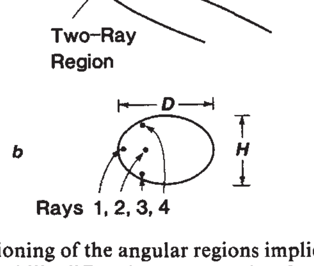

Fig. 5 *a*, Partitioning of the angular regions implied by the theory of hyperbolic-umbilic diffraction catastrophes for sections having $C_3 \neq 0$. *b*, Ray locations evident from the patches in Fig. 3. The camera's aperture was in the four-ray region.

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Rainbow scattering from spheroidal drops—an explanation of the hyperbolic umbilic foci

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An experiment by Marston and Trinh[1], showed how an oblate spheroidal water drop illuminated by a parallel beam of light forms a caustic in the far field characterized by the hyperbolic umbilic catastrophe—a generalization of the primary rainbow formed by a spherical drop. As the axis-ratio of the drop was changed, they observed a singular section of the catastrophe appearing for $D/H = 1.305 \pm 0.016$—D being the diameter in the horizontal equatorial plane and H the diameter along the vertical axis of rotational symmetry. I show here how this ratio may be calculated to confirm quantitatively the arguments[1] that the hyperbolic–umbilic catastrophe is associated with skew rays, and predict the occurrence of novel 'lips' phenomena, yet to be observed.

PQRQ'P' is the critical (Descartes) ray in the equatorial plane (Fig. 1). Keeping the incident direction fixed, if we were to move the entry point away from Q, but still in the equatorial plane, standard rainbow theory would then tell us that the resulting angle of deviation is a minimum for entry at Q—the angles of incidence, i, and refraction, r, are given by $\sin i = \surd\{(4-\mu^2)/3\}$, $\sin r = \surd\{(4-\mu^2)/3\mu^2\}$ and are connected by Snell's law $\sin i = \mu \sin r$, where μ is the refractive index.

If we now vary the entry point in the perpendicular (vertical) plane, still keeping the incident direction fixed, the once-refracted ray will, in general, pass above or below R by a first-order amount. However, we can show that, for a critical value of D/H, it continues to pass through R, to first order, and therefore emerges (by symmetry) displaced from the equatorial plane but without change of deviation, to first order. That is, we have directional focusing in the vertical as well as in the horizontal plane. Such simultaneous focusing in two directions gives rise to what is called, in catastrophe optics, an umbilic caustic. At R there is a horizontal focal line.

To find the critical D/H take the axes as shown, let the point of entry move vertically by ε, and assume that the refracted ray passes through R. To first order the direction cosines of the incident ray, the normal to the surface and the refracted ray are the rows of the determinant

$$\begin{vmatrix} \sin i & 0 & \cos i \\ 0 & \varepsilon/\rho & 1 \\ \sin r & \varepsilon/s & \cos r \end{vmatrix}$$

where $s = \mathrm{QR} = D \cos r$ and ρ is the radius of curvature of the surface in the vertical plane, $\frac{1}{2}H^2/D$.

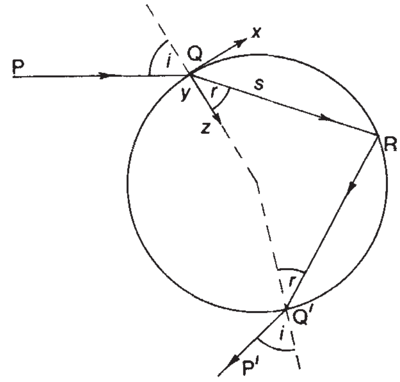

Fig. 1 The equatorial (horizontal) section of a spheroidal water drop showing the critical ray of minimum deviation.

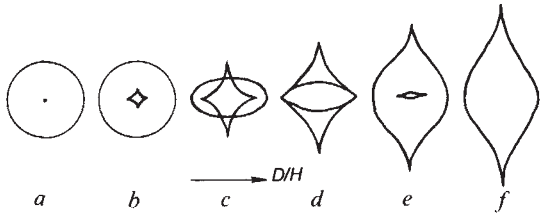

Fig. 2 Sequence of caustics in the far field as D/H increases from 1. *a*, The circular rainbow with a singular point at its centre. On perturbation the point breaks up (*b*, *c*) into an expanding four-cusped figure. At *d*, two hyperbolic umbilic foci occur. On further increase of D/H, the inner figure contracts (*e*), and then disappears (*f*) in a lips event. The angular width of the complete figure is the same throughout.

Snell's law is satisfied to first order without further conditions, but for the three directions to be coplanar the above determinant must be zero. A short calculation then yields the surprisingly simple result

$$\frac{D}{H} = \sec r = \sqrt{\frac{3\mu^2}{4(\mu^2-1)}}.$$

Putting $\mu = 1.3317$, the refractive index of water at the wavelength used in the experiments ($\lambda = 633$ nm), gives $D/H = 1.3114$, which agrees with the observed value of 1.305 ± 0.016. This is a stringent test that the shape of the drop is spheroidal.

I now show, using the principles of catastrophe optics, that the global topology of the caustic is that sketched in Fig. 2, taken from Hannay's work in appendix 2 of ref. 2. First notice that as $\mu \to 2$, the above critical ratio $D/H \to 1$ and that both i and r approach zero. That is, the caustic structure collapses into a single point in direction space (backwards)—the perfect 'cats-eye' reflector. Now unfold (that is, perturb) this degenerate focus by treating $(2-\mu)$ and D/H as control variables; keeping $D/H = 1$ and varying μ from 2 causes the familiar rainbow circle to grow from the central point (Fig 2*a*). Now keep μ fixed and increase D/H: the sequence is then Fig 2*a* to *f* with two hyperbolic umbilics appearing symmetrically in the equatorial plane in Fig 2*d*. These are the ones calculated above.

The lips event occurring between Fig. 2*e* and *f* is at $D/H = \surd\{\mu/2(\mu-1)\} = 1.4168$ for the above value of μ. (There appears to be a second horizontal lips event on the axis at $D/H = \surd\{(2\mu-1)/2(\mu-1)\} = 1.5835$, not part of this catastrophe and arising from rays that pass through the centre of vertical curvature of the far side of the drop.)

Equation (A2.2) in ref. 2 shows that the form of the wavefront that emerges from the drop in the most singular case (spherical drop with $\mu = 2$), specified by its deviation $f(x, y)$ from a plane, is, to lowest order, $f = A(x^4 + 2x^2y^2 + y^4)$, which is a singularity of infinite codimension; that is, there are an infinite number of unfolding terms. However, the symmetry-breaking perturbation