# Two-Dimensional Continuous Wavelet Analysis and Its Application to Meteorological Data


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# Outline

Introduction

2-D CWT

Common CWT Wavelets
- Morlet
- Halo and Arc
- Mexican Hat
- Cauchy
- Poisson
- Application Specific

Application to Synthetic Test Functions

Application to Meteorological Datasets

Conclusion

# Introduction

- In this paper, Wang and Lu demonstrate:
  - The important practical issues arising from the implementation and application of the 2D CWT
  - Pros and cons of different wavelet mother functions to help in selecting an appropriate mother function for a particular meteorological application
  - The power and limits of the 2D CWT by applying it to meteorological datasets

# 2-D CWT

$$f(\mathbf{x}) = C_\psi^{-1} \int_{[0,\infty]} \int_{[-\pi,\pi]} \int_{R^2} a^{-3} (Wf)(a,\mathbf{b},\theta) \psi_{a,\mathbf{b},\theta}(\mathbf{x}) \, d\mathbf{b} \, d\theta \, da \tag{5}$$

and

$$C_\psi = (2\pi)^2 \int_{R^2} |\hat{\psi}(\boldsymbol{\omega})|^2 |\boldsymbol{\omega}|^{-2} \, d\boldsymbol{\omega} < \infty. \tag{6}$$

## Wavelet Transform

- Important diagnostic tool for analyzing nonstationary signals

## Continuous Wavelet Transform (CWT)

- Continuously sweeping "microscope" for examining spectral components of datasets
- Integral transform of functions of $L^2(R)$ with a wavelet function as the transform kernel
- Applied to discrete dataset with continuously varying dilation and translation parameters
  - Translation parameter is a vector in the x-y plane
- CWT basis functions can be anywhere over the time/space frequency plane
- CWT produces a redundant set of info about the dataset, shifted invariant between different scales
- Provides detailed local spectral information for different scales about the field without between-scale shifting effects
- Spectral information about a signal for arbitrary scale *s*, orientation $\theta$, and physical location $(x,y)$
- Directional 2-D CWT with a rotation parameter enhances the ability to detect data singularity in a particular direction
- Important to select the appropriate mother function for a particular application

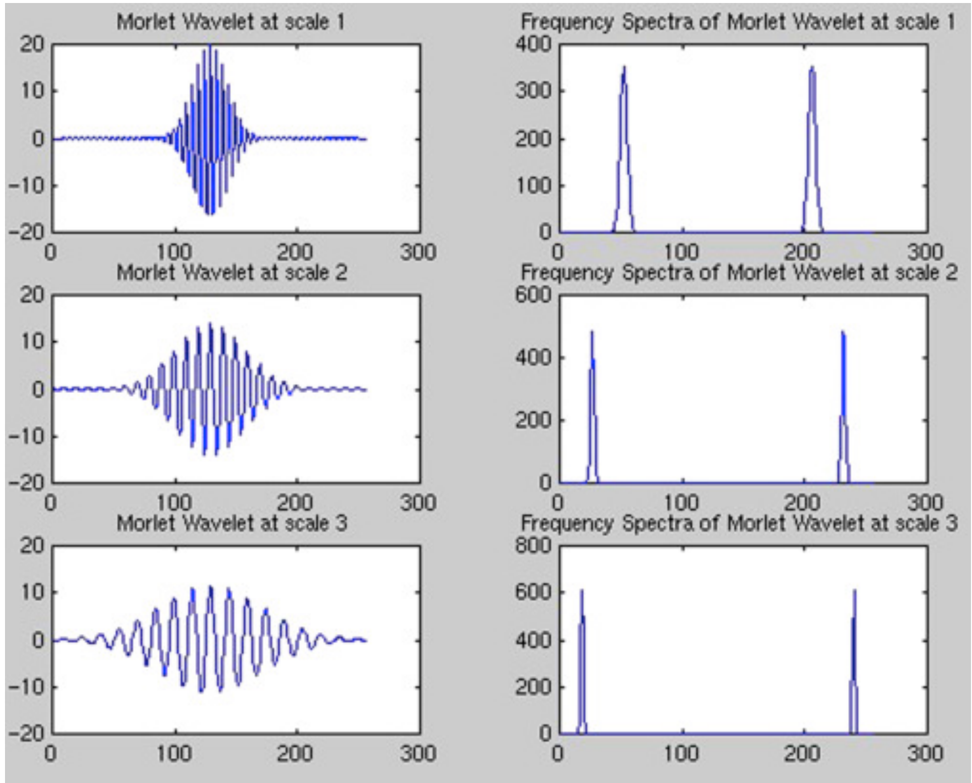
# Common CWT Wavelets

- Morlet Wavelet
- Halo and Arc Wavelets
- Mexican Hat Wavelet
- Cauchy Wavelet
- Poisson Wavelet
- Application specific wavelet

# Morlet Wavelet

$$\psi(\mathbf{x}) = e^{i\boldsymbol{\omega}_0 \cdot \mathbf{x}} e^{-(\epsilon^{-1}x_1^2 + x_2^2)/2} + \mathrm{corr}(\mathbf{x}). \qquad (10)$$

$$\hat{\psi}(\boldsymbol{\omega}) = \sqrt{\epsilon} e^{-[\epsilon(\omega_1 - \omega_{0_1})^2 + (\omega_2 - \omega_{0_2})^2]/2} \qquad (11)$$




**Most commonly used mother function**

**Two parameters**

- Wave vector $\omega_0$ – set to be a vector along y-axis
- Anisotropy parameter $\epsilon$ – controls the shape of the 2-D Gaussian window

**Features**

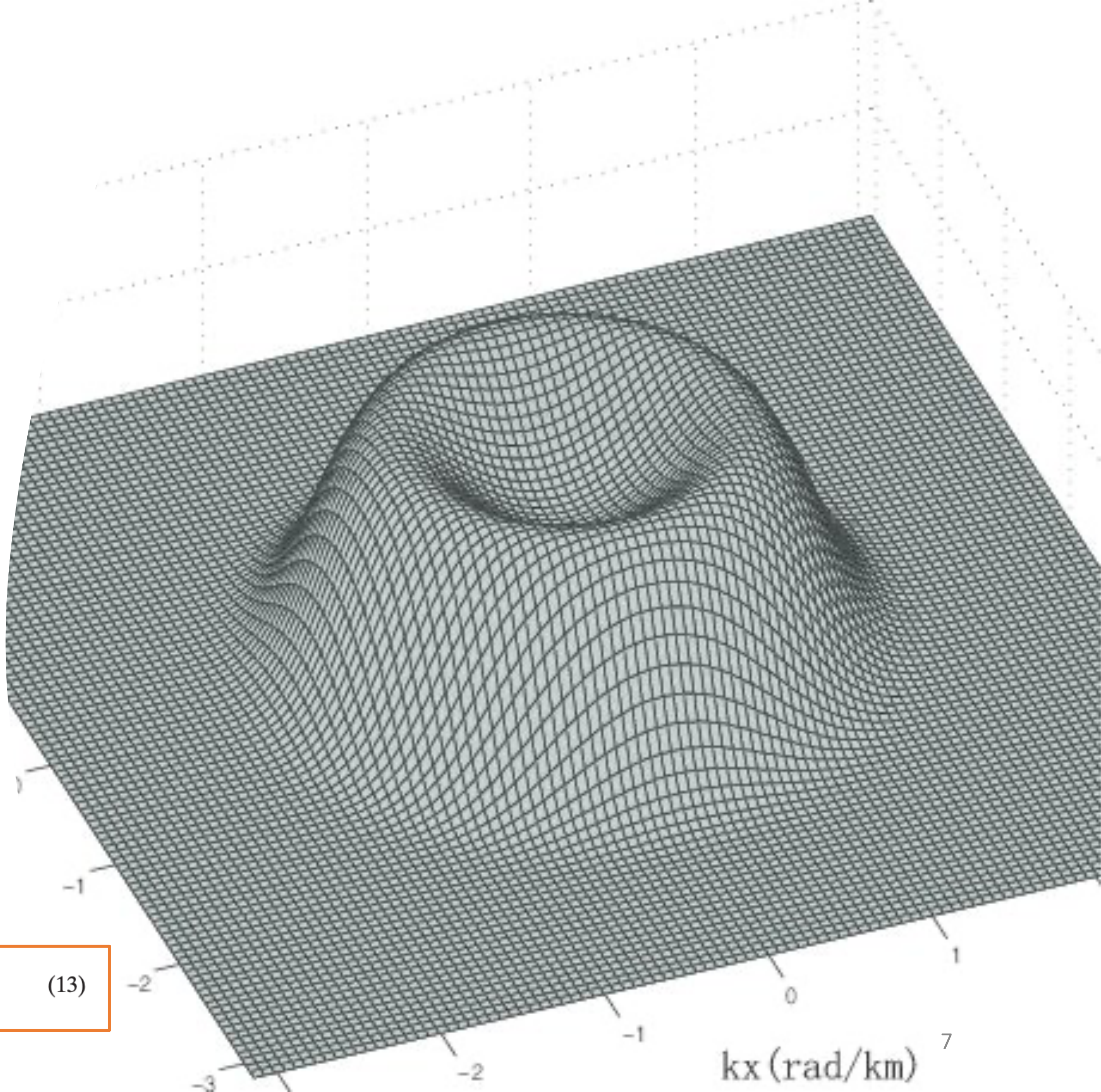
- Scale parameters are closely related to the Fourier wavelength
- Morlet transform provides phase information
- Good angular selectivity when appropriate parameters are selected
- Only reveals spectral information for a strip of a particular orientation
  - Either must apply Morlet wavelet transform over many azimuths or know the directions of interest

# Halo and Arc Wavelets

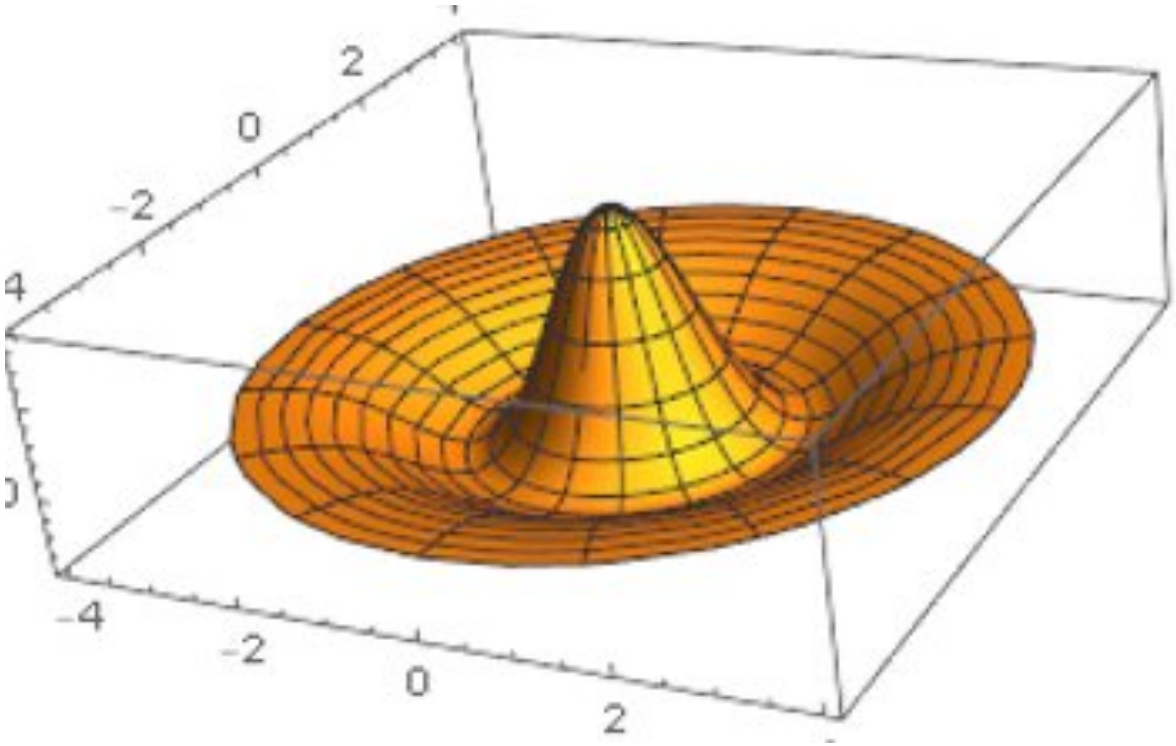
- Isotropic version of the morlet wavelet
- Removes the dimension of the azimuthal angle
- Spectrum retains the same close relationship with the Fourier spectrum
- Will not produce phase information
- Arc wavelet is a remedy to this problem

$$\widehat{\psi}(\boldsymbol{\omega}) = e^{-(|\boldsymbol{\omega}|-|\boldsymbol{\omega}_0|)^2/2}. \qquad (12)$$

$$\widehat{\psi}(\boldsymbol{\omega}) = \begin{cases} e^{-(|\boldsymbol{\omega}|-|\boldsymbol{\omega}_0|)^2/2} & \omega_2 \geq 0, \\ 0 & \text{otherwise.} \end{cases} \qquad (13)$$



# Mexican Hat Wavelet




- Real and isotropic
- Laplacian of the 2D Gaussian function
- Good local support in the physical domain
- Good wavelet for detecting edge contour features

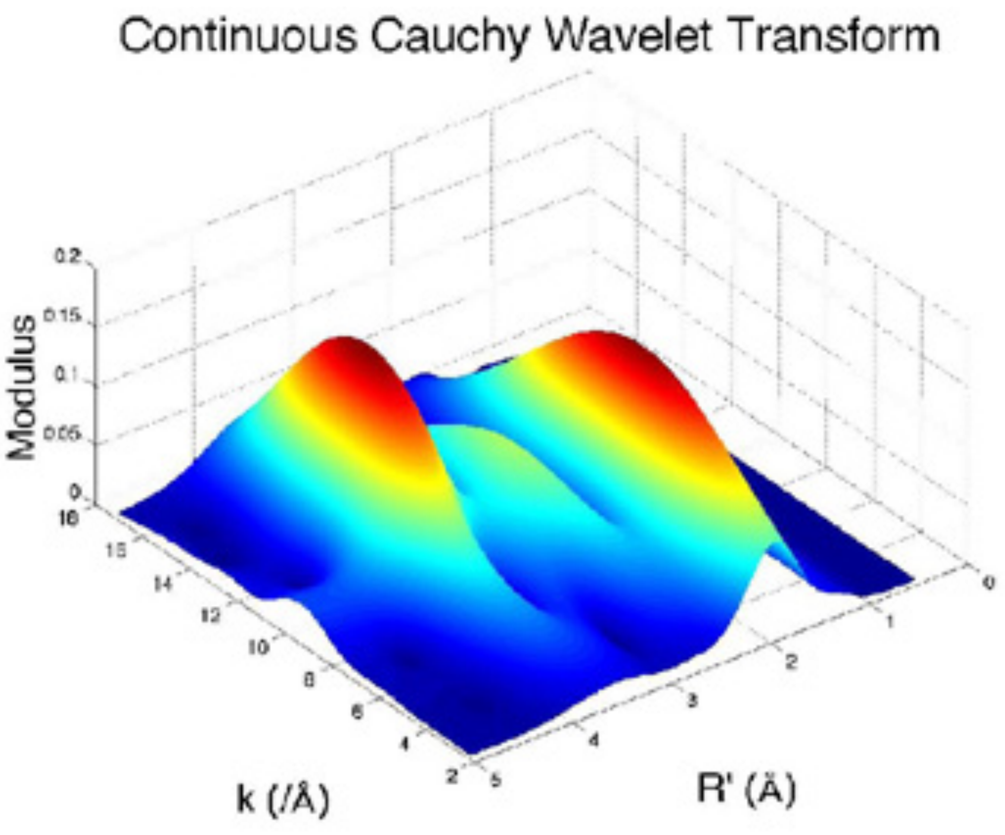
$$\psi(\mathbf{x}) = (2 - |\mathbf{x}|^2)e^{-|\mathbf{x}|^2/2}, \quad (14)$$

and its Fourier transform,

$$\widehat{\psi}(\boldsymbol{\omega}) = |\boldsymbol{\omega}|^2 e^{-|\boldsymbol{\omega}|^2/2}. \quad (15)$$

# Cauchy Wavelet

- 2-D directional anisotropic wavelet
- Has minimum uncertainty
- Similar to 2-D Morlet wavelet in angular selectivity and its ability to provide phase information




et al. 1999). Let $\mathcal{C}(\alpha, \beta) = \{\boldsymbol{\omega} \in R^2 | \alpha \leq \arg(\boldsymbol{\omega}) \leq \beta\}$, $\mathbf{e}_\theta = [\cos(\theta), \sin(\theta)]$, $\boldsymbol{\eta} = \mathbf{e}_{(\alpha+\beta)/2}$, and $m \in \mathbb{N}$:

$$\widehat{\psi}(\boldsymbol{\omega}) = \begin{cases} (\boldsymbol{\omega} \cdot \mathbf{e}_{\beta-\pi/2})^m (\boldsymbol{\omega} \cdot \mathbf{e}_{\alpha+\pi/2})^m e^{-\boldsymbol{\omega}\cdot\boldsymbol{\eta}} & \boldsymbol{\omega} \in \mathcal{C}(\alpha, \beta), \\ 0 & \text{otherwise.} \end{cases} \tag{16}$$

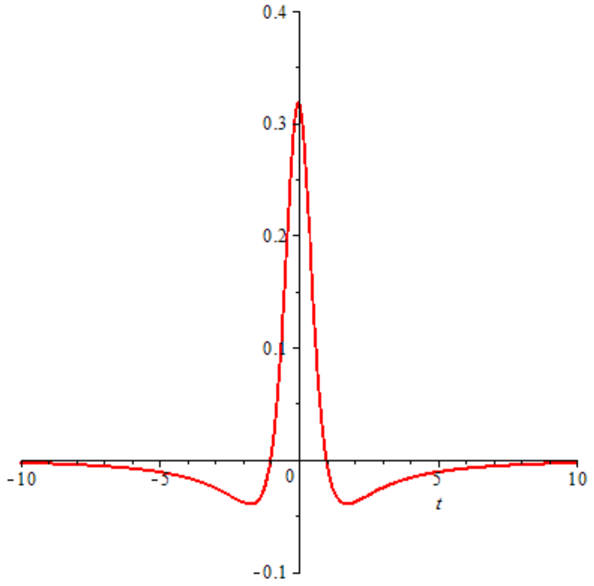
# Poisson Wavelet

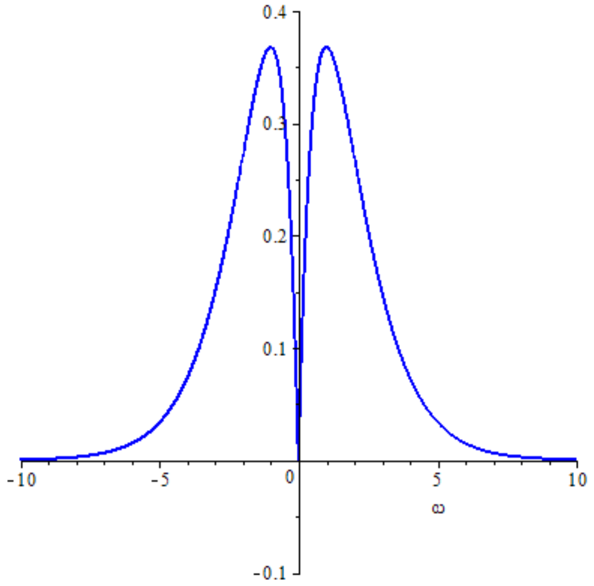
- Introduced to analyze the field of homogeneous point sources
- Used to retrieve the properties of the source from the measurement that is located in a distant hyperplane.

$$\psi(\mathbf{x}) = \frac{1 - |\mathbf{x}|^2}{(1 + |\mathbf{x}|^2)^2}, \qquad (17)$$

and its Fourier transform,

$$\widehat{\psi}(\boldsymbol{\omega}) = |\boldsymbol{\omega}|e^{-|\boldsymbol{\omega}|}. \qquad (18)$$



# Application-specific Wavelet

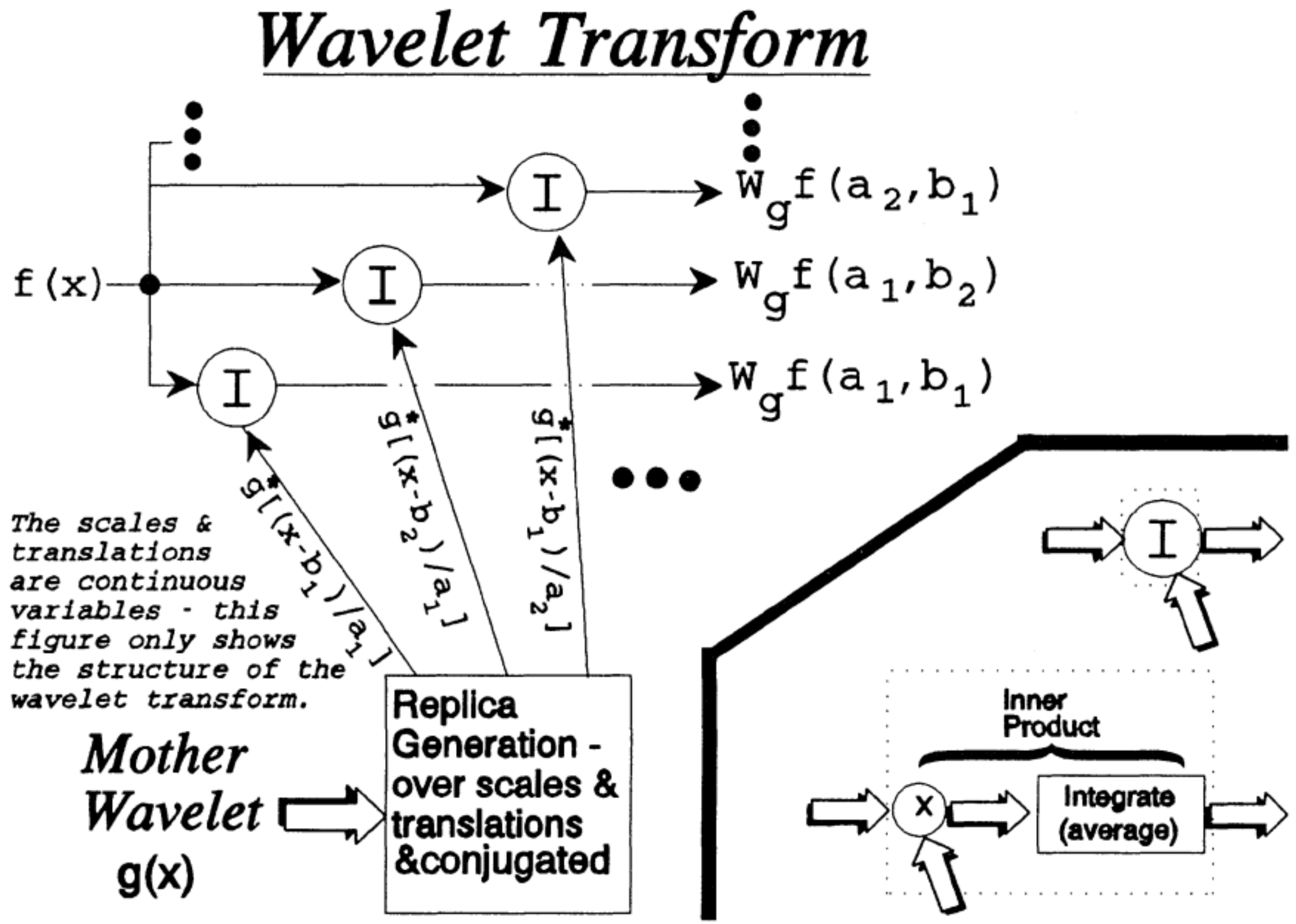


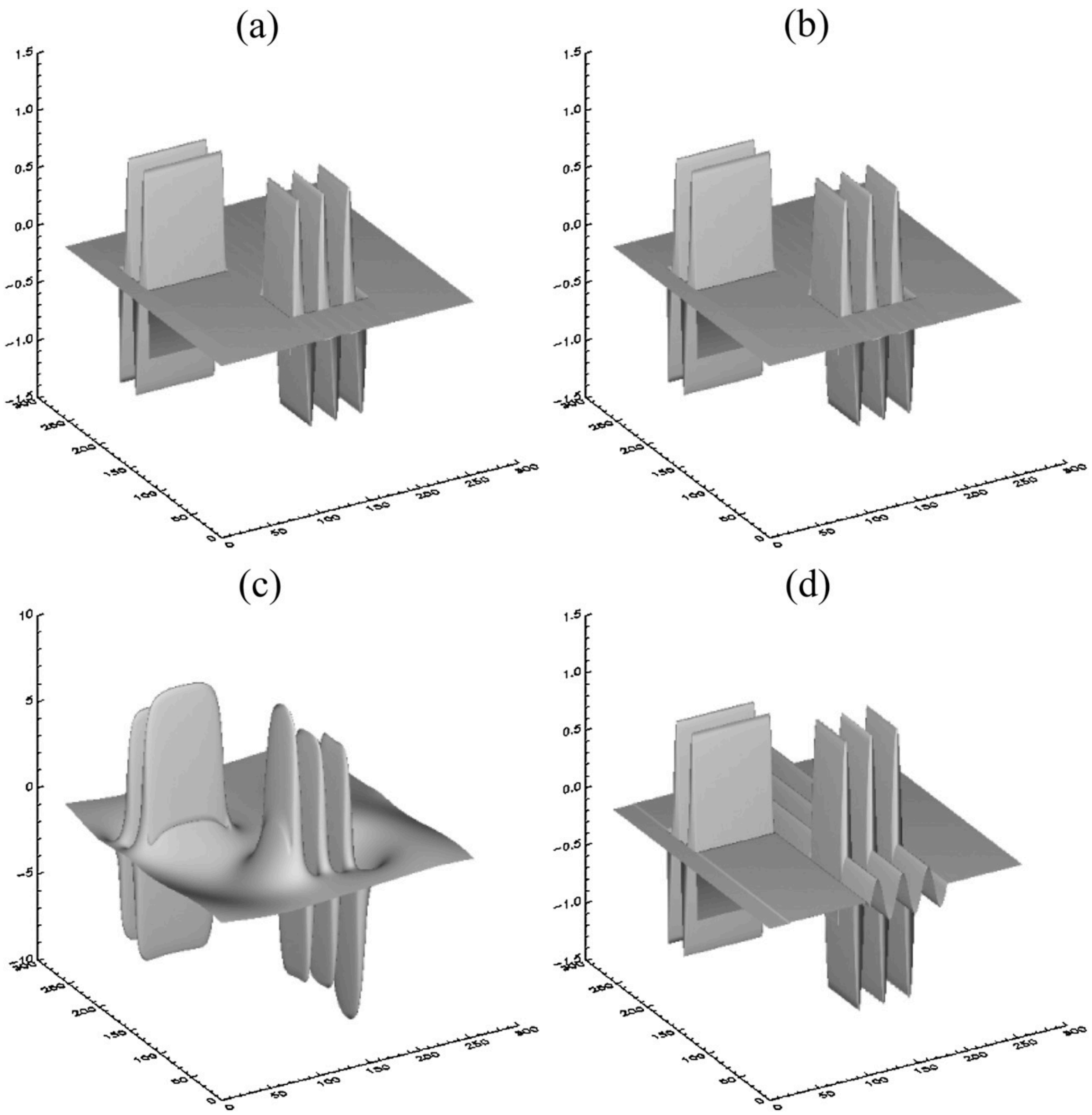

Figure 2.1: Wavelet Transform Structure

# Application to Synthetic Test Functions

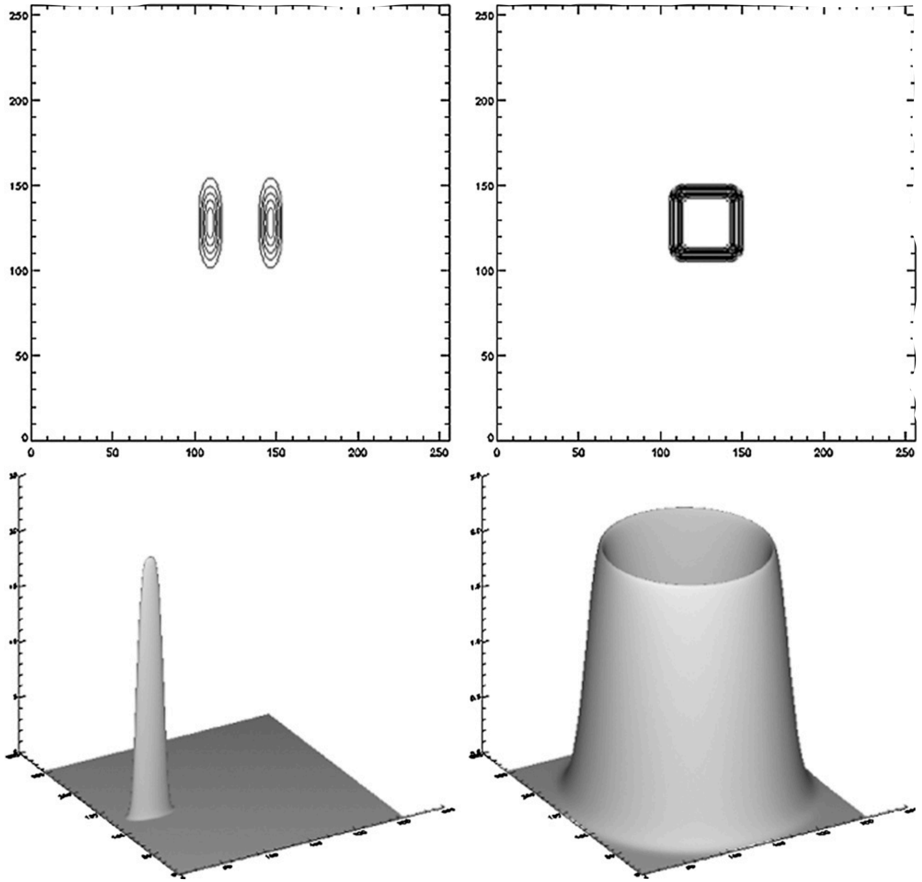
- Two rules for obtaining perfect reconstruction
  - The values of the scale parameter should form a geometric sequence, i.e. logarithmic scale discretization
  - When the wavelet in Fourier domain $\hat{\psi}(\omega)$ has its support in mostly half plane and/or is the superimposition of multiple anisotropic wavelet functions, $\hat{\psi}(\omega)$ needs to have the same overall frequency response along any direction
- The certainty of both spectral and location information is bounded by the Heisenberg uncertainty principle

# Reconstruction of a two-wave function

- (a) original function
- (b) perfectly reconstructed function
  - Superimposing wavelet coefficients of 40 scales with a constant interscale ratio
- (c) function synthesized with a set of wavelet coefficients from 200 linearly discretized scales without necessary additional scaling
  - Rule 1
- (d) function reconstructed from the coefficient sets of the wavelet transform whose basis function in the Fourier domain have an unnecessary duplicate along the $\omega_1$ axis
  - Rule 2



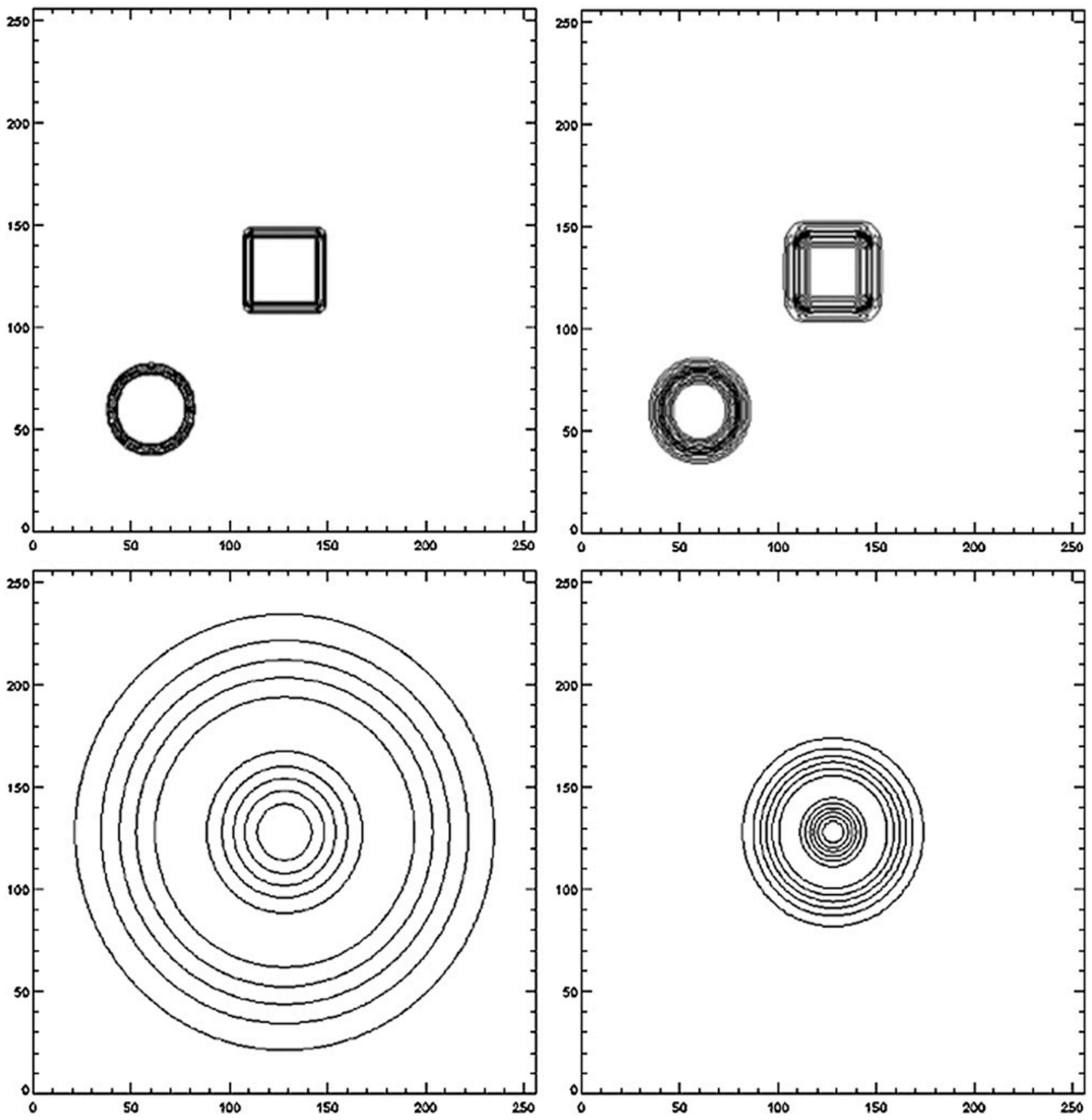
# Morlet and Halo transforms of a square function



- Morlet wavelet (left)
  - Anisotropic, scale 10, directed along the x-axis
  - Picks up wavelets along direction of orientation
- Halo wavelet (right)
  - Isotropic, scale 10
  - Picks up signals in all directions
- Wavelet basis functions in Fourier domain (bottom)
- Heisenberg uncertainty principle
  - “variances of any square-integrable function in physical and spectral domains cannot be arbitrarily small at the same time. When the function is normalized to have unit energy, the product of the two variances are bounded by a constant.”
  - The more we find out about physical location, the less certain we are about spectral location and vice versa

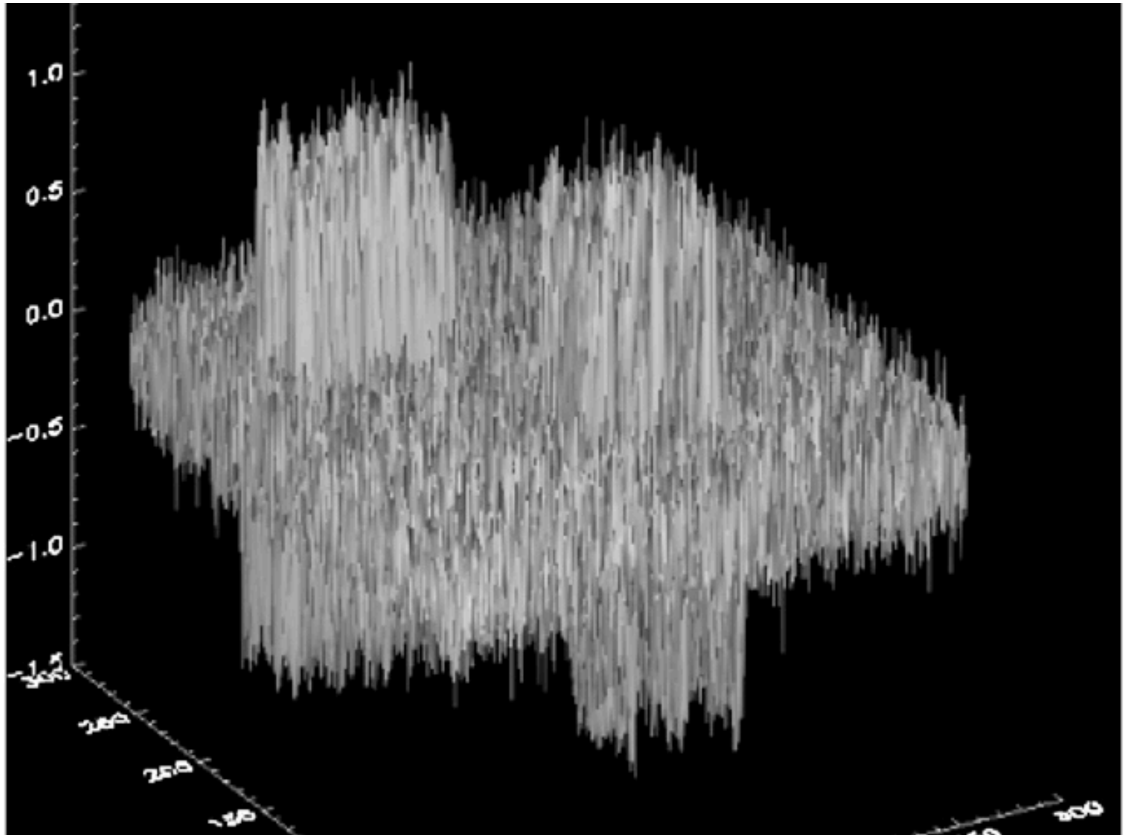
# Edge Detection

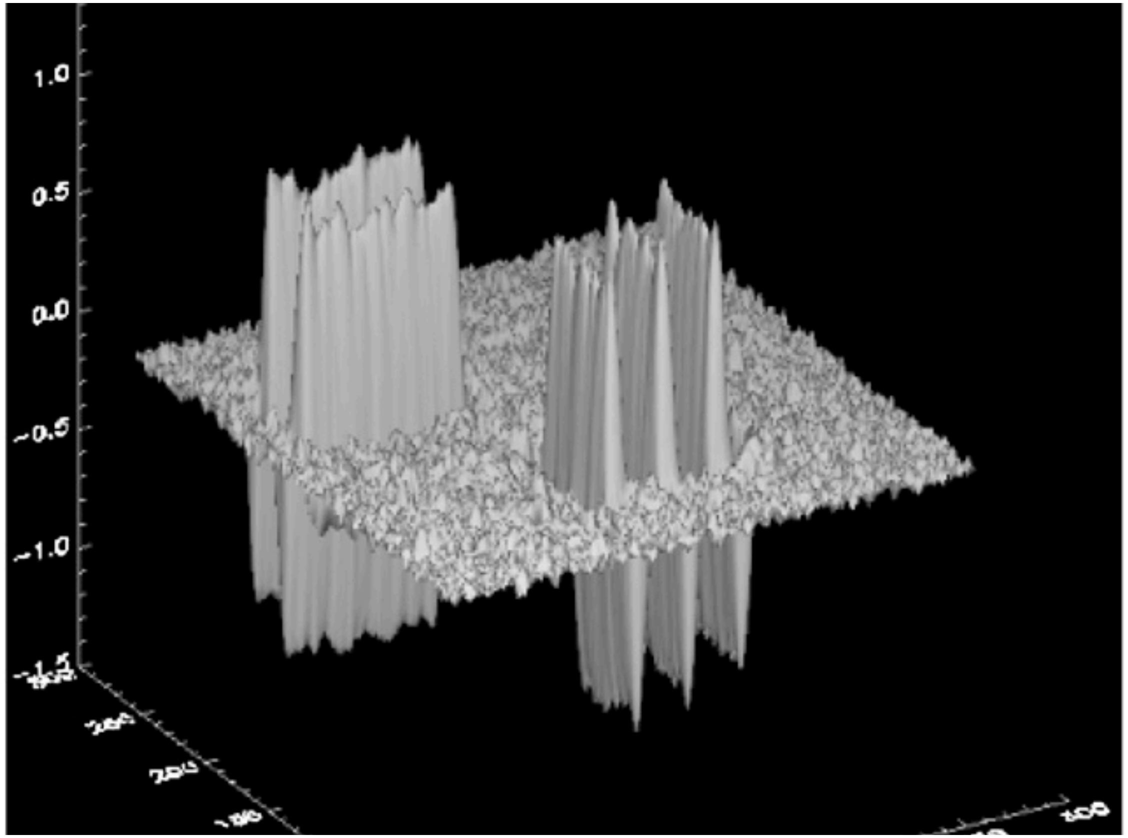
- Mexican hat wavelet
- Scales 10 (left) and 15 (right)
- At the smaller scale, the physical location of the edge is more precise
- At the smaller scale, in Fourier space, wavelet has slower decay, or less precise frequency response
- Not ideal for spectral information or frequency filtering



# Denoising

- Two perpendicular local waves imbedded in Gaussian noise
- Applied isotropic 2-D wavelet, suppressing coefficients of small scales



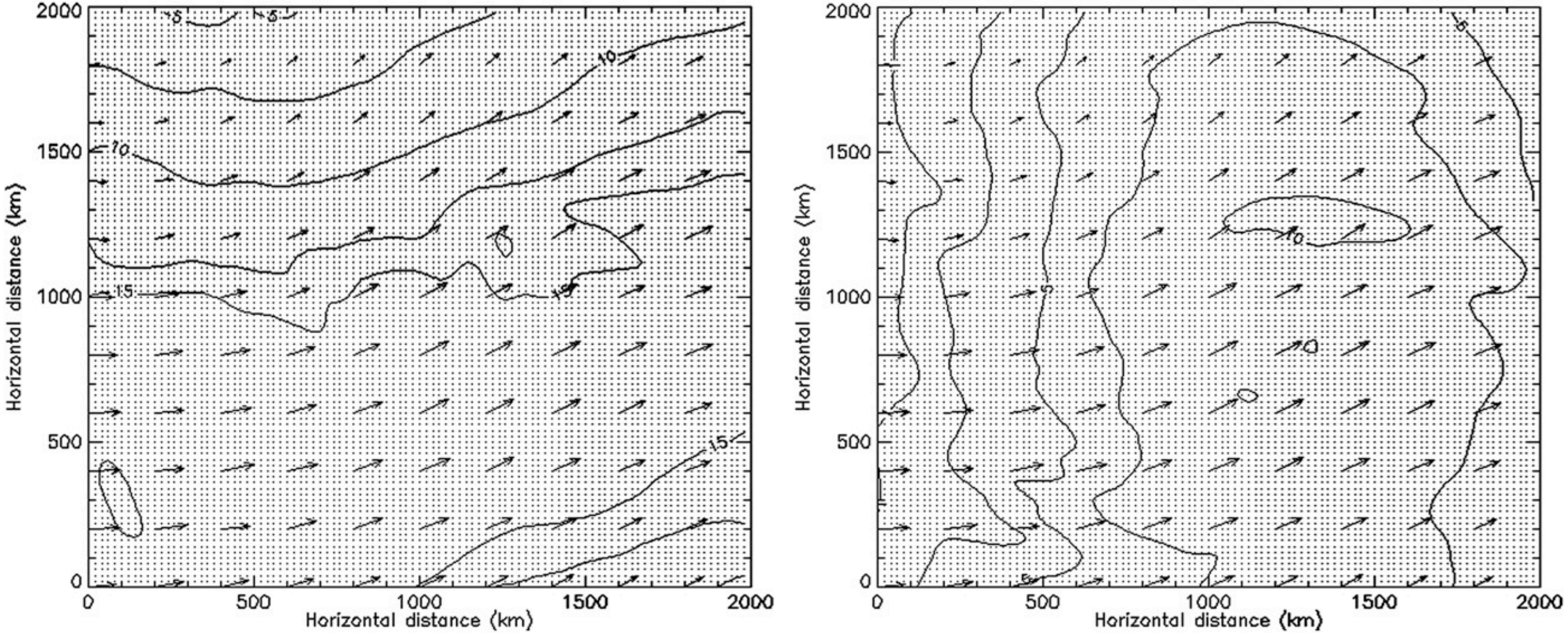
# Application to Meteorological Datasets

Two-dimensional scale decomposition

Field reconstruction

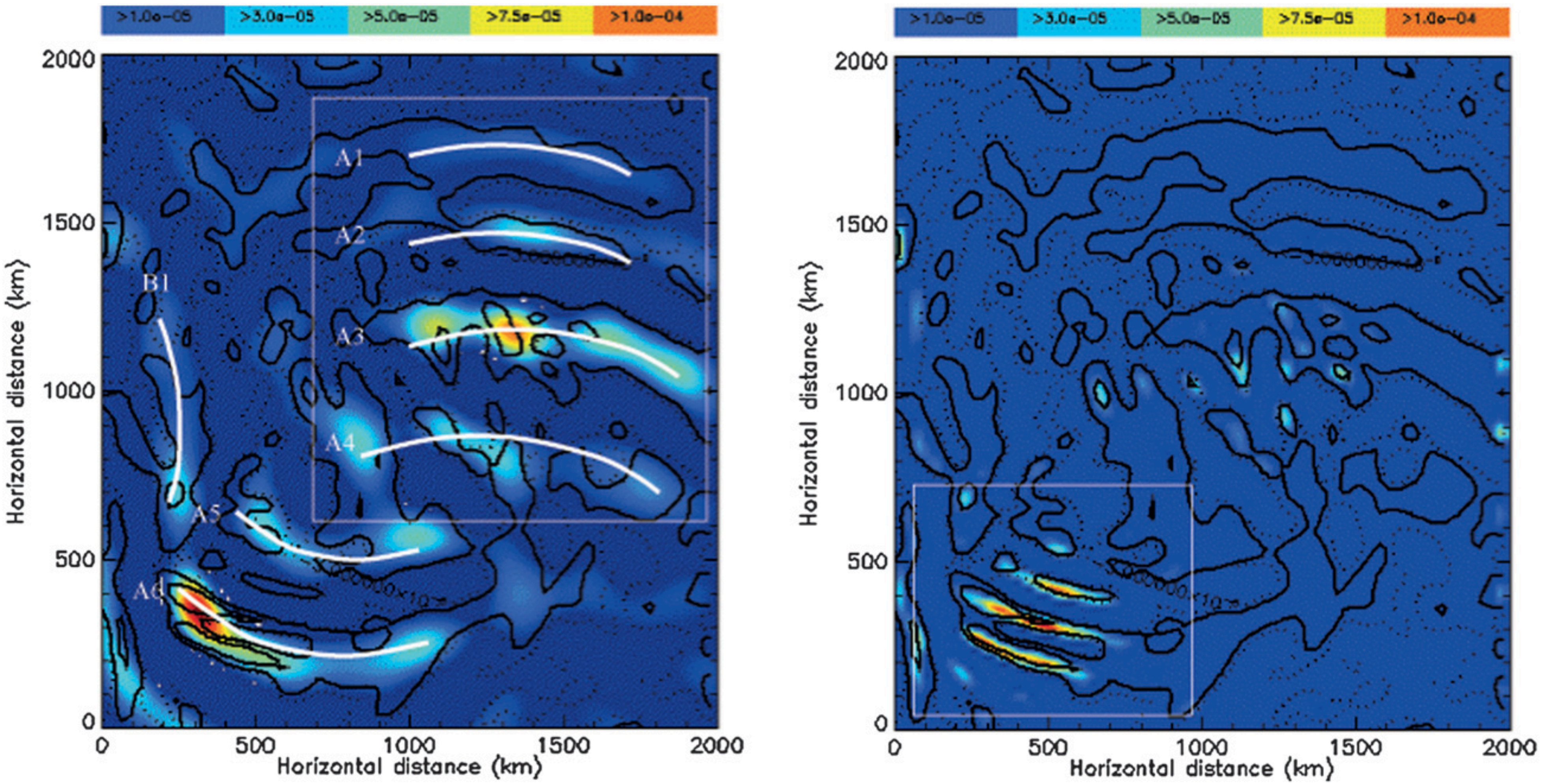
Feature localization

Directional signal detection

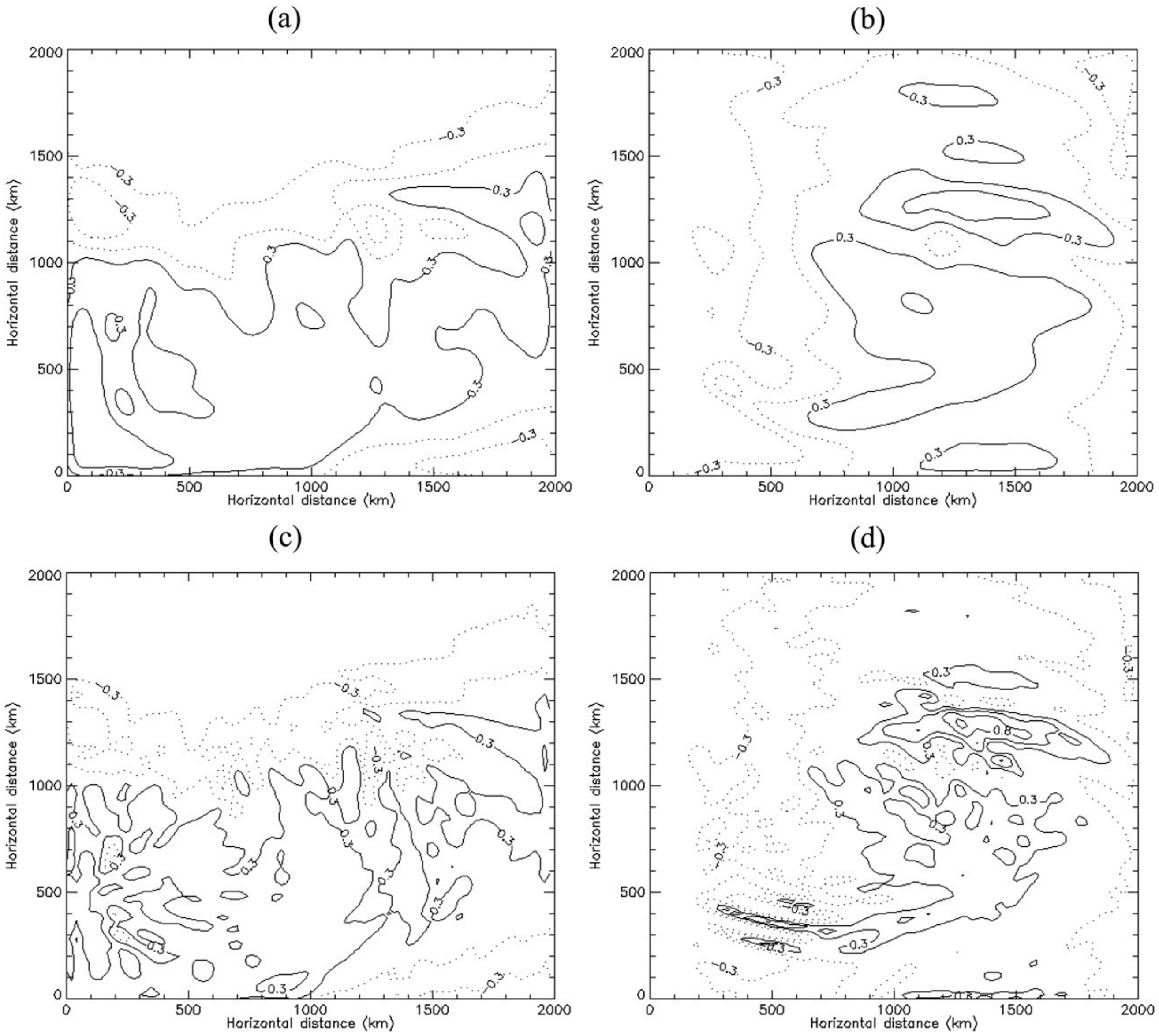


2000
1500
1000
500
0
Horizontal distance (km)
0
500
1000
1500
2000
Horizontal distance (km)
5
10
15

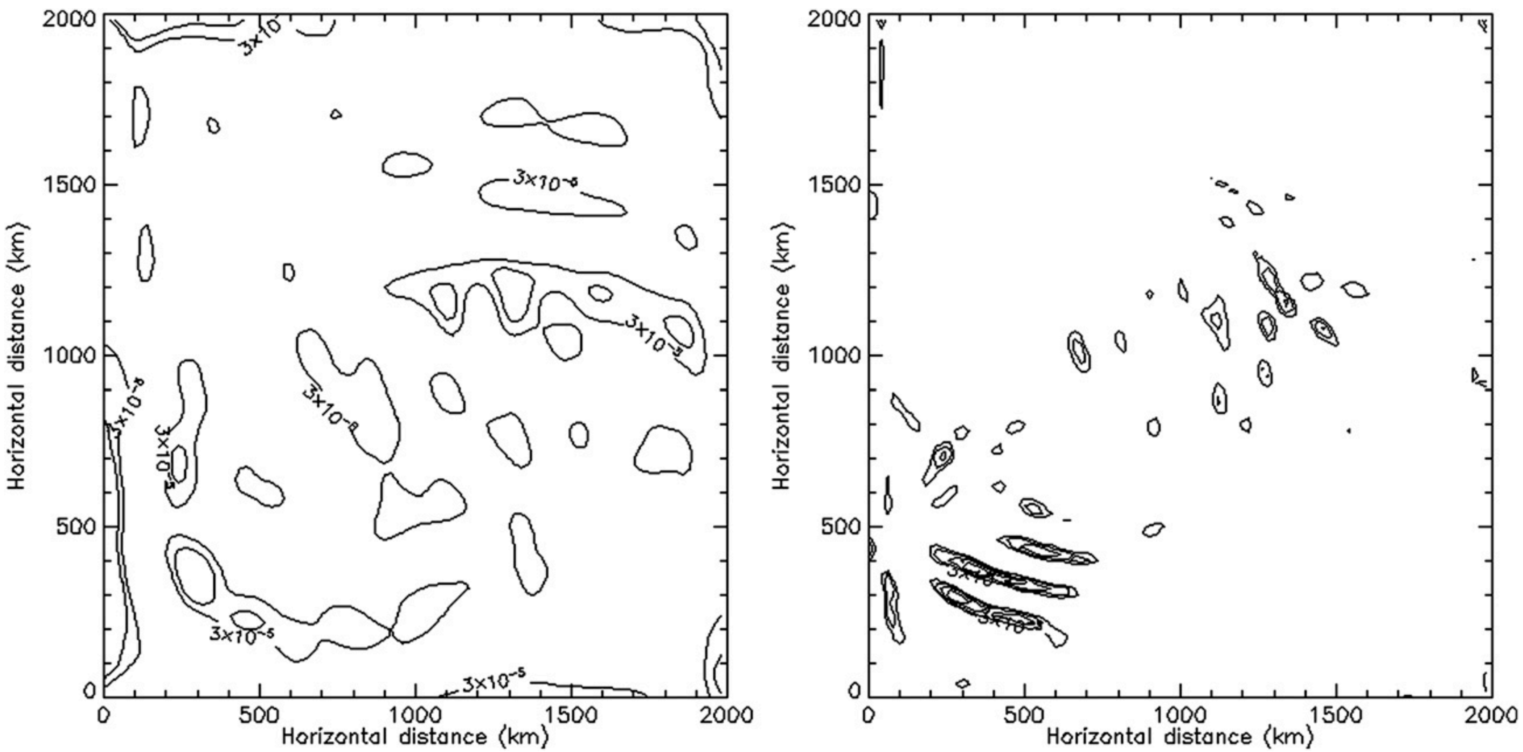
# Two-dimensional Scale Decomposition



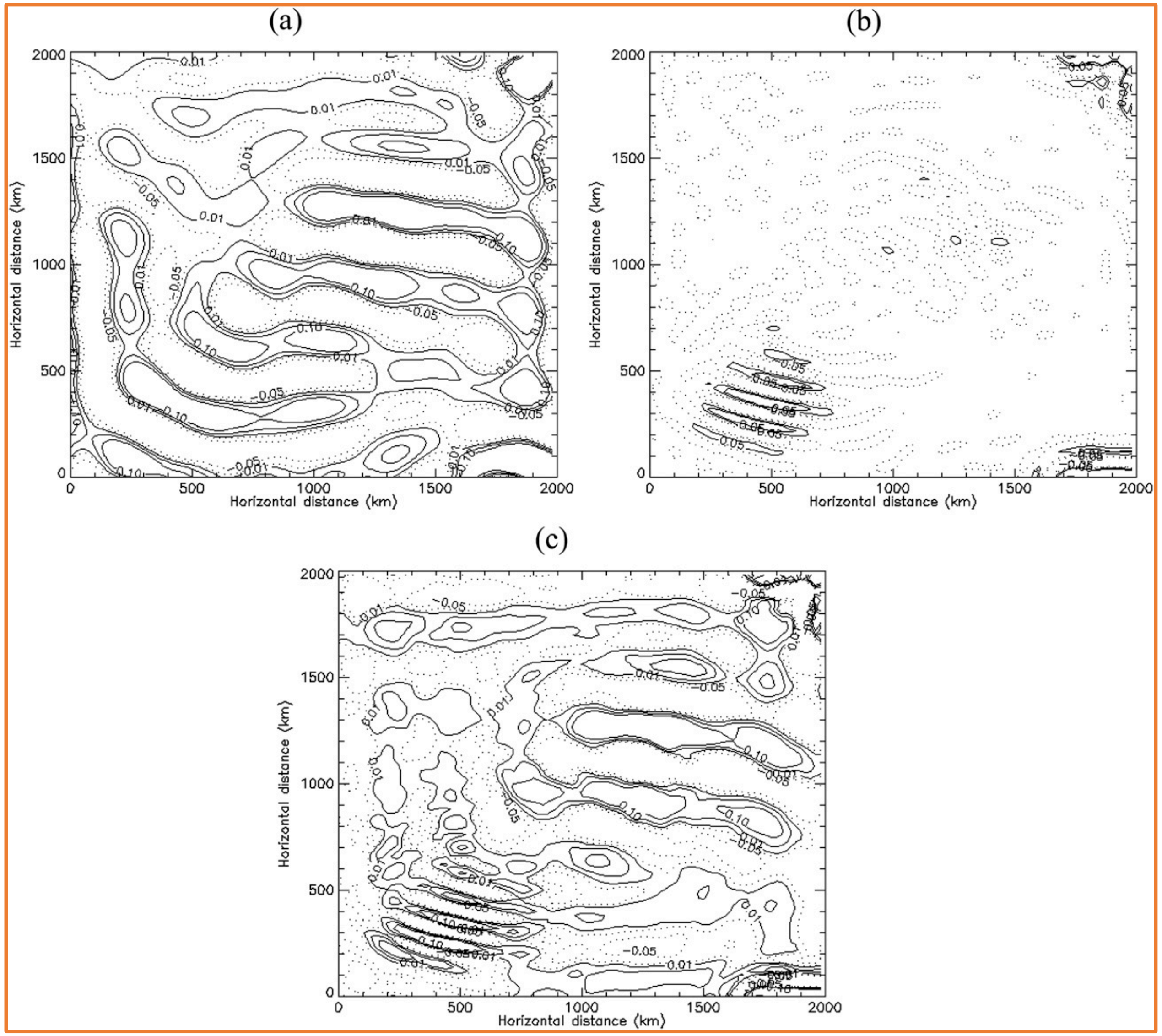
# Reconstructed wind components

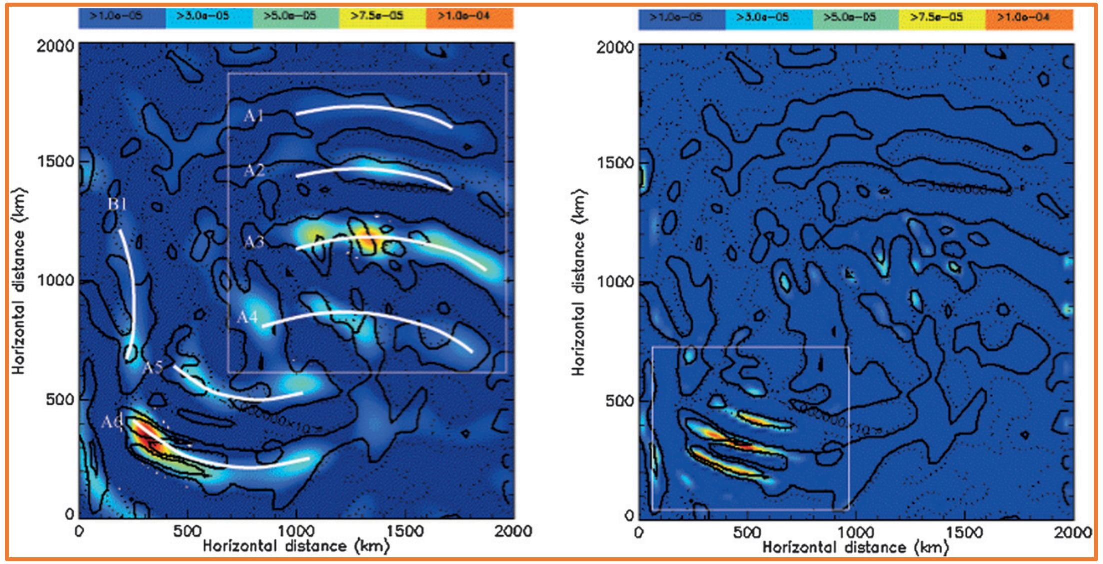


# Recomputed Divergence Field

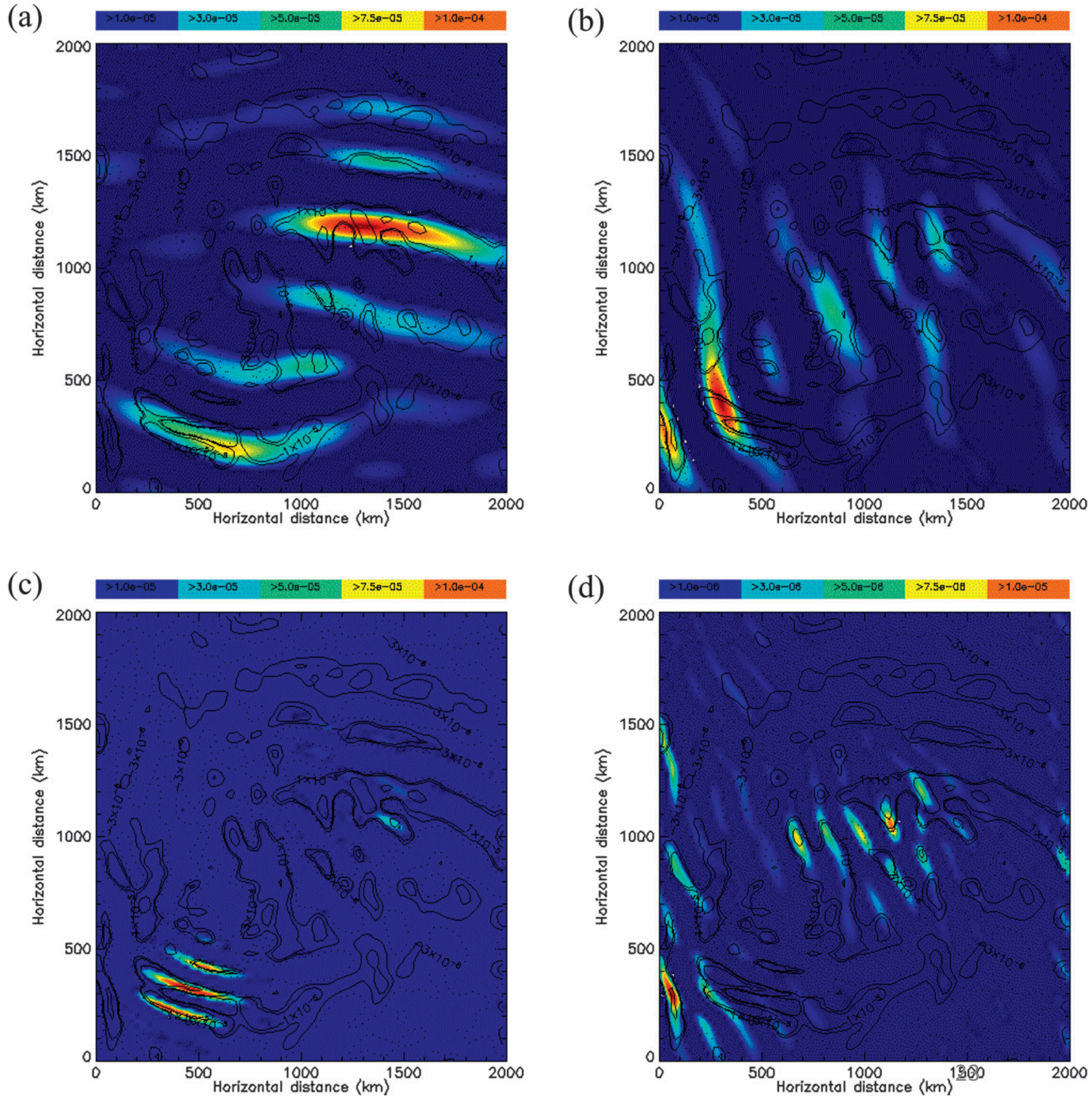


# Feature localization



# Directional wavelet analysis



# Conclusion

- 2D CWT provides comprehensive spectral information about the analyzed dataset at different physical locations and along different directions
- It is best to use isotropic wavelets to analyze multidimensional datasets to identify a direction of interest before applying anisotropic wavelet analysis
- Perfect reconstruction principle: analysis for a particular scale should only be drawn from the set of coefficients that perfectly reproduces the analyzed data
- Important to create objective criteria to automatically select scales for coherent structure analysis and feature detection

# Questions and Comments?